Introduction to
PLASMA ASTROPHYSICS
(Selected 10 lectures)

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About These Lectures

If you want to learn the most fundamental things about plasma astrophysics **with the least amount of time and effort** – and who doesn't? – this text is for you.

The text is addressed to students **without a background** in plasma physics.

It grew from the lectures given at the Moscow Institute of Physics and Technics (the 'fiz-tekh').

A similar full-year course was offered to the students of the Astronomical Division in the Faculty of Physics at the Moscow State University over after 1990.

The idea of these lectures is **not** typical for the majority of textbooks.

It was suggested by S.I. **Syrovatskii** that

> the **consecutive consideration** of physical principles, starting from the most general ones, and of simplifying assumptions **gives** us a simpler description of plasma under cosmic conditions.

On the basis of such an approach the student interested in modern astrophysics, its **current practice**, will find the answers to **two key questions**:

1. What approximation is **the best** one (the simplest but sufficient) for description of a phenomenon in astrophysical plasma?

2. **How** can I **build an adequate model** for the phenomenon, for example, a flare in the corona of an accretion disk?

Practice is really important for the theory of astrophysical plasma.

Related exercises (supplemented to each chapter) serve to better understanding of plasma astrophysics.

As for the **applications**, preference evidently is given to physical processes in the **solar plasma**.

Why? – Because of the possibility of the all-round observational **test** of theoretical models.

For instance, **flares on the Sun**, in contrast to those on other stars, **can be seen** in their development.

We can obtain a sequence of **images** during the flare's evolution, not only in the optical and radio ranges but also in the EUV, soft and hard X-ray, **gamma-ray** ranges.

For beginning students, who may not know in which subfields of astrophysics they wish to specialize,

> it is better to cover a lot of fundamental theories **thoroughly** than to dig deeply into any particular astrophysical subject or object,

even a very interesting one, for example black holes.

Astrophysicists of the future will need **tools** that allow them to explore in many different directions.

Moreover astronomy of the future will be, more than hitherto, **precise science** similar to mathematics and physics.

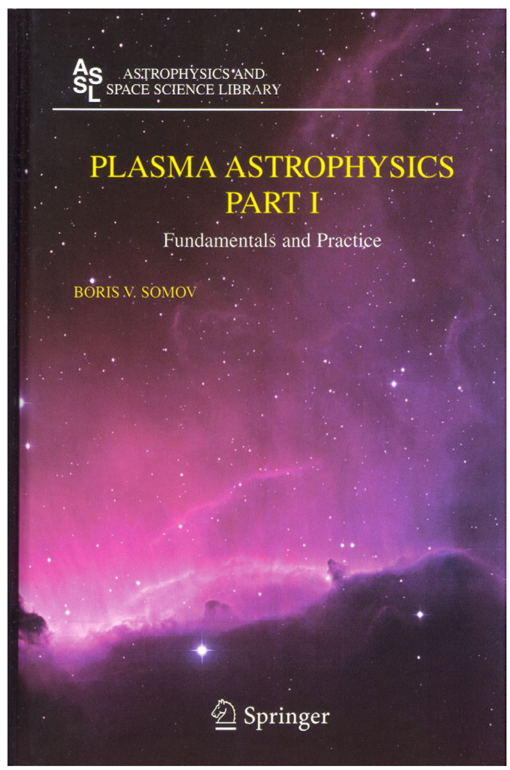


Figure 1: The first volume of the book covers the **basic principles and main practical tools** required for work in plasma astrophysics.

see http://www.springer/com/

http://adsabs.harvard.edu/

The second volume “Plasma Astrophysics, Part II, Reconnection and Flares” represents the basic physics of the magnetic **reconnection effect and the flares** of electromagnetic origin in the solar system, relativistic objects, accretion disks, their coronae.

Never say: “It is easy to show...”.

Chapter 1

Particles and Fields: Exact Self-Consistent Description

There exist **two ways** to describe **exactly** the behaviour of a system of charged particles in electromagnetic and gravitational fields.

1.1 Liouville's theorem

1.1.1 Continuity in phase space

Let us consider a system of N **interacting particle**.

Without much justification, let us introduce the distribution function

$$f = f(\mathbf{r}, \mathbf{v}, t) \tag{1.1}$$

for particles as follows.

We consider the **six-dimensional** (6D) space called **phase space** $X = \{\, \mathbf{r}, \mathbf{v} \}$ shown in Fig. 1.1.

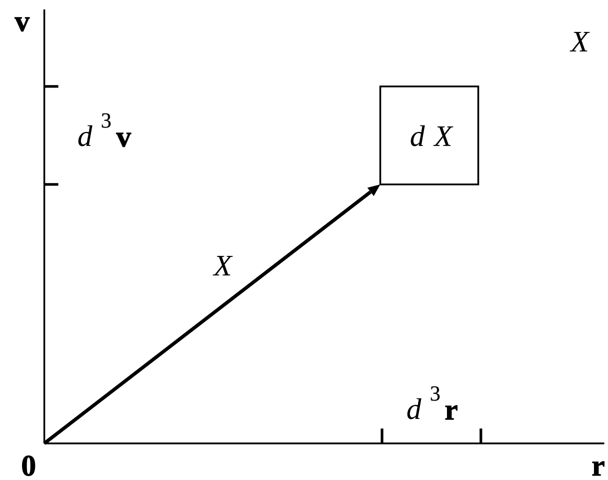


Figure 1.1: The 6D phase space X. A small volume dX at a point X.

The number of particles present in a **small volume** $dX = d^3\mathbf{r}\, d^3\mathbf{v}$ at a point X at a moment of time t is defined to be

$$dN(X,t) = f(X,t)\, dX. \tag{1.2}$$

Accordingly, the **total number** of the particles at this moment is

$$N(t) = \int f(X,t)\, dX \equiv \iint f(\mathbf{r},\mathbf{v},t)\, d^3\mathbf{r}\, d^3\mathbf{v}\,. \tag{1.3}$$

If, for definiteness, we use the Cartesian coordinates, then

$$X = \{\, x,\ y,\ z,\ v_x,\ v_y,\ v_z \,\}$$

is a **point** of the phase space (Fig. 1.2) and

$$\dot{X} = \{\, v_x,\, v_y,\, v_z,\, \dot{v}_x,\, \dot{v}_y,\, \dot{v}_z \,\} \tag{1.4}$$

is the **velocity** of this point in the phase space.

Suppose the coordinates and velocities of the particles are changing **continuously** – 'from point to point', i.e. the **particles move smoothly at all times**.

So the distribution function $f(X, t)$ is **differentiable**.

Moreover we assume that this motion of the particles in phase space can be expressed by the **continuity equation**:

$$\boxed{\frac{\partial f}{\partial t} + \mathrm{div}_X\, f\dot{X} = 0} \tag{1.5}$$

or

$$\frac{\partial f}{\partial t} + \mathrm{div}_{\mathbf{r}}\, f\mathbf{v} + \mathrm{div}_{\mathbf{v}}\, f\dot{\mathbf{v}} = 0\,.$$

Equation (1.5) expresses the **conservation law** for the particles, since the integration of (1.5) over a volume U enclosed by the surface S in Fig. 1.2 gives

$$\frac{\partial}{\partial t}\int\limits_U f\, dX + \int\limits_U \mathrm{div}_X\, f\dot{X}\, dX =$$

by virtue of the **Ostrogradskii-Gauss** theorem

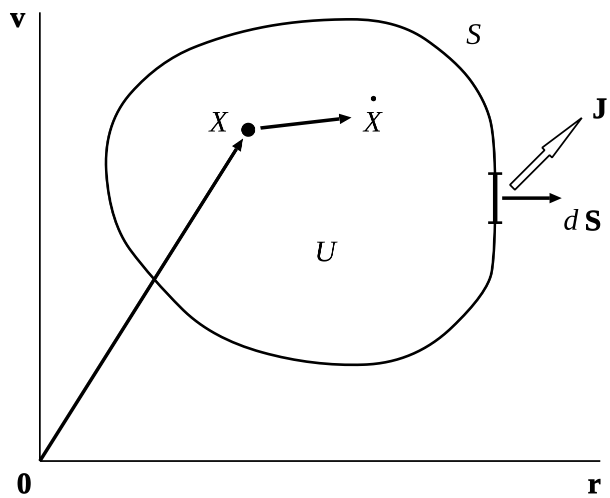


Figure 1.2: The 6D phase space X. The volume U is enclosed by the surface S.

$$= \frac{\partial}{\partial t} N(t)\Big|_U + \int_S f\dot{X}\, dS = \frac{\partial}{\partial t} N(t)\Big|_U + \int_S \mathbf{J}\cdot d\,\mathbf{S} = 0\,. \quad (1.6)$$

Here

$$\mathbf{J} = f\dot{X} \quad (1.7)$$

is the **particle flux density** in the phase space.

Thus

> a change of the particle number in a given volume U of the phase space X is defined by the **particle flux** through the boundary surface S.

The reason is clear.

There are **no sources or sinks** for the particles inside the volume.

Otherwise the source and sink terms must be added to the right-hand side of Equation (1.5).

1.1.2 The character of particle interactions

Let us rewrite Equation (1.5) in another form in order to understand the **meaning of divergent terms**.

The first of them is

$$\mathrm{div}_{\mathbf{r}}\, f\mathbf{v} = f\ \mathrm{div}_{\mathbf{r}}\, \mathbf{v} + (\mathbf{v} \cdot \nabla_{\mathbf{r}})\, f = \ 0 + (\mathbf{v} \cdot \nabla_{\mathbf{r}})\, f\,,$$

since $\mathbf{r}$ and $\mathbf{v}$ are independent variables in the phase space X.

The second divergent term is

$$\mathrm{div}_{\mathbf{v}}\, f\dot{\mathbf{v}} = f\ \mathrm{div}_{\mathbf{v}}\, \dot{\mathbf{v}} + \dot{\mathbf{v}} \cdot \nabla_{\mathbf{v}}\, f\,.$$

So far no assumption has been made as to the **character of particle interactions**.
It is worth doing here.

Let us restrict our consideration to the interactions with

$$\boxed{\mathrm{div}_{\mathbf{v}}\, \dot{\mathbf{v}} = 0\,,} \tag{1.8}$$

then Equation (1.5) takes the following form

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla_{\mathbf{r}}\, f + \frac{\mathbf{F}}{m} \cdot \nabla_{\mathbf{v}}\, f = 0$$

or

$$\frac{\partial f}{\partial t} + \dot{X}\, \nabla_X\, f = 0\,, \tag{1.9}$$

where

$$\dot{X} = \left\{ v_x,\, v_y,\, v_z,\, \frac{F_x}{m}\,,\, \frac{F_y}{m}\,,\, \frac{F_z}{m} \right\}. \tag{1.10}$$

So we 'trace' the **phase trajectories** of particles when they move under action of a force field $\mathbf{F}(\mathbf{r}, \mathbf{v}, t)$.

Thus we have found Liouville's theorem in the following formulation:

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla_{\mathbf{r}} f + \frac{\mathbf{F}}{m} \cdot \nabla_{\mathbf{v}} f = 0\,. \tag{1.11}$$

> **Liouville's theorem**: *The distribution function remains constant on the particle phase trajectories* **if condition** (1.8) **is satisfied**.

We call Equation (1.11) the **Liouville equation**.

The first term in Equation (1.11), the **partial** time derivative $\partial f/\partial t$, characterizes a change of the distribution function $f(t, X)$ at a given point X in the phase space with time t.

Define also the **Liouville operator**

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \dot{X}\, \frac{\partial}{\partial X} \equiv \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla_{\mathbf{r}} + \frac{\mathbf{F}}{m} \cdot \nabla_{\mathbf{v}}\,. \tag{1.12}$$

This operator is just the **total** time derivative following a particle motion in the phase space X.

By using definition (1.12), we rewrite Liouville's theorem as follows:

$$\boxed{\frac{Df}{Dt} = 0\,.} \tag{1.13}$$

What factors do lead to the changes in the distribution function?

Let dX be a small volume in the phase space X.

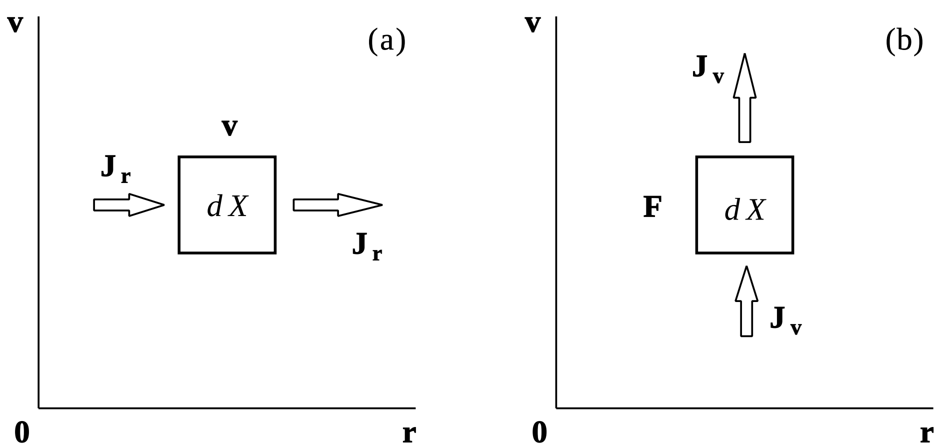


Figure 1.3: Action of the two different terms of the Liouville operator in the 6D space X.

The second term in (1.11), $\mathbf{v} \cdot \nabla_{\mathbf{r}} f$, means that the particles come into and go out of the volume element dX because their **velocities are not zero** (Fig. 1.3a).

So this term describes a **simple kinematic effect**.

> If the distribution function f has a gradient over $\mathbf{r}$, then a number of particles inside the volume dX changes because **they move** with velocity $\mathbf{v}$.

The third term, $(\mathbf{F}/m) \cdot \nabla_{\mathbf{v}} f$, means that the particles escape from the volume element dX or come into it due to their **acceleration** or **deceleration** under action of the force field $\mathbf{F}$ (Fig. 1.3b).

1.1.3 The Lorentz force, gravity

In order the Liouville theorem to be valid, the force field $\mathbf{F}$ has to satisfy **condition** (1.8).

We rewrite it as follows:

$$\frac{\partial \dot{v}_\alpha}{\partial v_\alpha} = \frac{1}{m} \frac{\partial F_\alpha}{\partial v_\alpha} = 0$$

or

$$\frac{\partial F_\alpha}{\partial v_\alpha} = 0 \,, \quad \alpha = 1, 2, 3 \,. \tag{1.14}$$

In particular, this condition holds if

> the component F_α of the force vector $\mathbf{F}$ does **not** depend upon the velocity component v_α.

This is a sufficient condition, of course.

The classical **Lorentz force**

$$F_\alpha = e \left[E_\alpha + \frac{1}{c} \, (\, \mathbf{v} \times \mathbf{B} \,)_\alpha \right] \tag{1.15}$$

obviously has that property.

The **gravitational force** in the classical approximation is entirely independent of velocity.

Other forces are considered, depending on a situation, e.g., the force resulting from the emission of radiation (the **radiation reaction**) and/or absorption of radiation by astrophysical plasma.

These forces when they are important must be considered with account of their relative significance, conservative or **dissipative** character, and other physical properties taken.

1.1.4 Collisional friction

As a contrary example we consider the collisional **drag force** which acts on a particle moving with velocity $\mathbf{v}$ in plasma:

$$\mathbf{F} = - \, k \, \mathbf{v} \, , \tag{1.16}$$

where the constant $k > 0$.

In this case the right-hand side of Liouville's equation is not zero:

$$-f\ \mathrm{div}_{\mathbf{v}}\,\dot{\mathbf{v}} = -f\ \mathrm{div}_{\mathbf{v}}\,\frac{\mathbf{F}}{m} = \frac{3k}{m}\,f\,,$$

because

$$\frac{\partial\, v_\alpha}{\partial\, v_\alpha} = \delta_{\alpha\alpha} = 3\,.$$

Instead of Liouville's equation we have

$$\frac{Df}{Dt} = \frac{3k}{m}\,f\ >\ 0\,. \tag{1.17}$$

Thus the distribution function (i.e. the particle density) does **not** remain constant on particle trajectories but increases with time.

Along the phase trajectories, it increases **exponentially**:

$$f(t,\mathbf{r},\mathbf{v}) \sim f(0,\mathbf{r},\mathbf{v})\ \exp\left(\frac{3k}{m}\,t\right). \tag{1.18}$$

The physical sense of this phenomenon is obvious.

The friction force **decelerates** the particles.

They go down in Fig. 1.4 and are concentrated in the vicinity of the axis $\mathbf{v} = \mathbf{0}$.

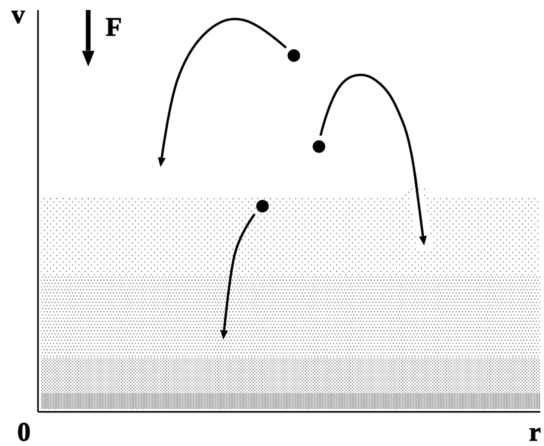


Figure 1.4: Particle density increases in the phase space as a result of action of the friction force $\mathbf{F}$.

1.1.5 The exact distribution function

Let us consider another property of the Liouville theorem.

We introduce the N-particle distribution function of the form

$$\hat{f}(t, \mathbf{r}, \mathbf{v}) = \sum_{i=1}^{N} \delta\left(\mathbf{r} - \mathbf{r}_i(t)\right) \, \delta\left(\mathbf{v} - \mathbf{v}_i(t)\right). \tag{1.19}$$

The delta function of the vector-argument is defined as usually:

$$\delta\left(\mathbf{r} - \mathbf{r}_i(t)\right) = \prod_{\alpha=1}^{3} \delta_\alpha =$$

$$= \delta\left(r_x - r_x^i(t)\right) \, \delta\left(r_y - r_y^i(t)\right) \, \delta\left(r_z - r_z^i(t)\right). \tag{1.20}$$

We shall call function (1.19) the **exact** distribution function.

It is illustrated by Fig. 1.5.

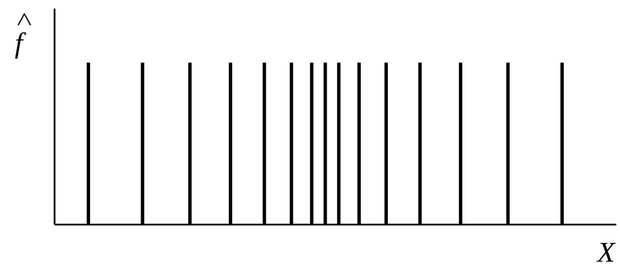


Figure 1.5: The one-dimensional analogy of the exact distribution function.

Let us substitute the exact distribution function in the Liouville equation.

Action:

$$\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla_{\mathbf{r}} + \frac{\mathbf{F}}{m}\cdot\nabla_{\mathbf{v}} \quad ==> \quad \hat{f} \quad ==> \quad = 0\,.$$

The resulting three terms are

$$\frac{\partial \hat{f}}{\partial t} = \sum_i (-1)\ \delta'_\alpha\left(\mathbf{r}-\mathbf{r}_i(t)\right)\ \dot{r}^{\,i}_\alpha\ \delta\left(\mathbf{v}-\mathbf{v}_i(t)\right) +$$

$$+\sum_i (-1)\ \delta\left(\mathbf{r}-\mathbf{r}_i(t)\right)\ \delta'_\alpha\left(\mathbf{v}-\mathbf{v}_i(t)\right)\ \dot{v}^{\,i}_\alpha\ , \qquad (1.21)$$

$$\mathbf{v}\cdot\nabla_{\mathbf{r}}\,\hat{f} \equiv v_\alpha\,\frac{\partial \hat{f}}{\partial r_\alpha} =$$

$$= \sum_i v_\alpha \, \delta'_\alpha \left(\mathbf{r} - \mathbf{r}_i(t) \right) \; \delta \left(\mathbf{v} - \mathbf{v}_i(t) \right) \,, \tag{1.22}$$

$$\frac{\mathbf{F}}{m} \cdot \nabla_{\mathbf{v}} \hat{f} \equiv \frac{F_\alpha}{m} \frac{\partial \hat{f}}{\partial v_\alpha} =$$

$$= \sum_i \frac{F_\alpha}{m_i} \, \delta \left(\mathbf{r} - \mathbf{r}_i(t) \right) \; \delta'_\alpha \left(\mathbf{v} - \mathbf{v}_i(t) \right) \,. \tag{1.23}$$

Here the index $\alpha = 1, 2, 3$ or (x, y, z).

The **prime** denotes the derivative with respect to the argument of a function.

The **overdot** denotes differentiation with respect to time t.

Summation over the **repeated index** α (**contraction**) is implied:

$$\delta'_\alpha \, \dot{r}^{\,i}_\alpha = \delta'_x \, \dot{r}^{\,i}_x + \delta'_y \, \dot{r}^{\,i}_y + \delta'_z \, \dot{r}^{\,i}_z \,.$$

The sum of terms (1.21)–(1.23) equals zero.

Let us rewrite it as follows

$$0 = \sum_i \left(-\dot{r}^{\,i}_\alpha + v^i_\alpha \right) \, \delta'_\alpha \left(\mathbf{r} - \mathbf{r}_i(t) \right) \, \delta \left(\mathbf{v} - \mathbf{v}_i(t) \right) +$$

$$+ \sum_i \left(-\dot{v}^{\,i}_\alpha + \frac{F_\alpha}{m_i} \right) \, \delta \left(\mathbf{r} - \mathbf{r}_i(t) \right) \, \delta'_\alpha \left(\mathbf{v} - \mathbf{v}_i(t) \right) \,.$$

This can occur just then that **all the coefficients** of different combinations of delta functions with their derivatives **equal zero** as well.

Therefore we find

$$\frac{d\,r_\alpha^{\,i}}{dt} = v_\alpha^{\,i}(t)\,, \qquad \frac{d\,v_\alpha^{\,i}}{dt} = \frac{1}{m_i}\,F_\alpha\left(\mathbf{r}_i(t), \mathbf{v}_i(t)\right)\,. \tag{1.24}$$

Thus

> the Liouville equation for an exact distribution function is **equivalent** to the Newton set of equations for a particle motion, both describing a purely **dynamic** behavior of the particles.

It is natural since this distribution function is **exact**. **No statistical averaging has been done so far**.

Statistics will appear later on when, instead of the **exact description** of a system, we begin to use some mean characteristics such as temperature, density etc.

The **statistical description** is valid for systems containing a large number of particles.

We have shown that finding a solution of the **Liouville equation for an exact distribution function**

$$\boxed{\frac{D\hat{f}}{Dt} = 0} \tag{1.25}$$

is the same as the integration of the motion equations.

However

> for systems of a **large number** of interacting particles, it is much more advantageous to deal with the **single** Liouville equation for the exact distribution function which describes the entire system.

1.2 Charged particles in the electromagnetic field

1.2.1 General formulation of the problem

Let us recall the basic physics notations and establish a common basis.

Maxwell's equations for the electric field **E** and magnetic field **B** are well known to have the form:

$$\operatorname{rot}\mathbf{B} = \frac{4\pi}{c}\,\mathbf{j} + \frac{1}{c}\,\frac{\partial\,\mathbf{E}}{\partial t}\,, \tag{1.26}$$

$$\operatorname{rot}\mathbf{E} = -\frac{1}{c}\,\frac{\partial\,\mathbf{B}}{\partial t}\,, \tag{1.27}$$

$$\operatorname{div}\mathbf{B} = 0\,, \tag{1.28}$$

$$\operatorname{div}\mathbf{E} = 4\pi\rho^{\,\mathrm{q}}\,. \tag{1.29}$$

> The fields are completely determined by electric charges and electric currents.

Note that Maxwell's equations imply:

- the **continuity equation for electric charge** (see Exercise 1.5)
- the **conservation** law for electromagnetic **field energy** (Exercise 1.6).

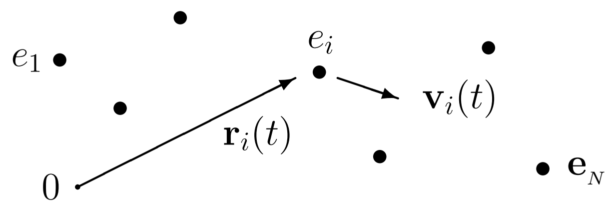


Figure 1.6: A system of N charged particles.

Let there be N particles with charges e_1, e_2, $\ldots$ e_i, $\ldots e_N$, coordinates $\mathbf{r}_i(t)$ and velocities $\mathbf{v}_i(t)$, see Fig. 1.6.

By definition, the **electric charge** density

$$\rho^{\,\mathrm{q}}(\mathbf{r}, t) = \sum_{i=1}^{N} e_i\,\delta\left(\mathbf{r} - \mathbf{r}_i(t)\right) \qquad (1.30)$$

and the density of **electric current**

$$\mathbf{j}\,(\mathbf{r}, t) = \sum_{i=1}^{N} e_i\,\mathbf{v}_i(t)\,\delta\left(\mathbf{r} - \mathbf{r}_i(t)\right). \qquad (1.31)$$

The coordinates and velocities of particles can be found by integrating the equations of motion – the Newton equations:

$$\dot{\mathbf{r}}_i = \mathbf{v}_i(t)\,, \tag{1.32}$$

$$\dot{\mathbf{v}}_i = \frac{1}{m_i}\, e_i \left[\mathbf{E}\left(\mathbf{r}_i(t)\right) + \frac{1}{c}\, \mathbf{v}_i \times \mathbf{B}\left(\mathbf{r}_i(t)\right) \right]. \tag{1.33}$$

Let us count the **number of unknown quantities**: the vectors $\mathbf{B}$, $\mathbf{E}$, $\mathbf{r}_i$, and $\mathbf{v}_i$.

We obtain: $3 + 3 + 3N + 3N = 6\,(N + 1)$.

The **number of equations** $= 8 + 6N = 6\,(N + 1) + 2$.

Therefore **two equations seem to be unnecessary**. Why is this so?

1.2.2 The continuity equation for electric charge

At first let us make sure that the definitions (1.30) and (1.31) conform to the **conservation law for electric charge**.

Differentiating (1.30) with respect to time gives

$$\frac{\partial \rho^{\mathrm{q}}}{\partial t} = - \sum_i e_i\, \delta'_\alpha\, \dot{r}^{\,i}_\alpha\,. \tag{1.34}$$

Here the index $\alpha = 1, 2, 3$.

The prime denotes the derivative with respect to the argument of the delta function.

The overdot denotes differentiation with respect to time t.

For the electric current density (1.31) we have the divergence

$$\operatorname{div}\mathbf{j} = \frac{\partial}{\partial r_\alpha}\, j_\alpha = \sum_i e_i\, v^i_\alpha\, \delta'_\alpha\,. \tag{1.35}$$

Comparing (1.34) with (1.35) we see that

$$\boxed{\frac{\partial \rho^{\mathrm{q}}}{\partial t} + \operatorname{div}\mathbf{j} = 0\,.} \tag{1.36}$$

Therefore the definitions for ρ^{q} and $\mathbf{j}$ conform to the continuity equation.

As we shall see it in Exercise 1.5, conservation of electric charge follows also directly from the Maxwell equations.

The difference is that above we have not used **scalar** Equation (1.29).

1.2.3 Initial equations and initial conditions

Operating with the divergence on Equation (1.26)

Action:

$$\operatorname{div} \quad ==> \quad \operatorname{rot}\mathbf{B} = \frac{4\pi}{c}\mathbf{j} + \frac{1}{c}\,\frac{\partial\,\mathbf{E}}{\partial t}\,,$$

and using the continuity Equation (1.36),

Action:

$$\operatorname{div} \mathbf{j} = -\frac{\partial \rho^{\mathrm{q}}}{\partial t}.$$

we obtain

$$0 = \frac{4\pi}{c}\left(-\frac{\partial \rho^{\mathrm{q}}}{\partial t}\right) + \frac{1}{c}\frac{\partial}{\partial t}\operatorname{div} \mathbf{E}.$$

Thus, we find that

$$\frac{\partial}{\partial t}\left(\operatorname{div} \mathbf{E} - 4\pi\rho^{\mathrm{q}}\right) = 0. \tag{1.37}$$

Hence Equation (1.29) will be valid at any moment of time, **provided** it is true at the initial moment.

Let us operate with the divergence on Equation (1.27):

Action:

$$\operatorname{div} \quad ==> \quad \operatorname{rot} \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t},$$

$$\frac{\partial}{\partial t}\operatorname{div} \mathbf{B} = 0. \tag{1.38}$$

Equation (1.28) implies the absence of magnetic charges or, which is the same, the **solenoidal** character of the magnetic field.

Conclusion. Equations (1.28) and (1.29) play the role of **initial conditions** for the time-dependent equations

$$\frac{\partial}{\partial t}\mathbf{B} = -c\operatorname{rot} \mathbf{E} \tag{1.39}$$

and

$$\frac{\partial}{\partial t}\,\mathbf{E} = +\,c\,\mathrm{rot}\,\mathbf{B} - 4\pi\,\mathbf{j}\,. \tag{1.40}$$

Thus, in order to describe the gas consisting of N charged particles, we consider the time-dependent problem of N bodies with a given **interaction law**.

> The electromagnetic part of interaction is described by Maxwell's equations, the time-independent **scalar equations playing the role of initial conditions** for the time-dependent problem.

Therefore the set consisting of eight Maxwell's equations and $6N$ Newton's equations is **neither over- nor under-determined**.

It is **closed** with respect to the time-dependent problem, i.e. it consists of $6\,(N+1)$ equations for $6\,(N+1)$ variables, once the initial and boundary conditions are given.

1.2.4 Astrophysical plasma applications

The set of equations described above can be treated analytically in just **three cases**:

1. $N = 1$, the motion of a charged particle in a **given** electromagnetic field, e.g., drift motions and adiabatic invariants, wave-particle interaction, **particle acceleration** in astrophysical plasma.

2. $N = 2$, Coulomb collisions of two charged particles, i.e. **binary collisions**.

   This is important for the kinetic description of physical processes, e.g., the **kinetic effects** under propagation of accelerated particles in plasma, collisional heating of plasma by a **beam** of fast electrons or/and ions.

3. $N \to \infty$, a very **large number** of particles.

   This case is the frequently considered one in plasma astrophysics, because it allows us to introduce **macroscopic descriptions** of plasma, the widely-used magnetohydrodynamic (MHD) approximation.

Intermediate case:

Numerical integration of Equations (1.26)–(1.33) in the case of **large but finite** N, like $N \approx 3 \times 10^6$, is possible by using modern computers.

The computations called **particle simulations** are increasingly useful for understanding many properties of astrophysical plasma and for **demonstration** of them.

One important example of a simulation is **magnetic reconnection** in a collisionless plasma.

This process often leads to fast energy conversion from field energy to particle energy, **flares** in astrophysical plasma (see Part II).

Generalizations:

The set of equations described can be generalized to include consideration of **neutral** particles.

This is necessary, for instance, in the study of the **generalized Ohm's law** which is applied in the investigation of physical processes in **weakly-ionized** plasmas, e.g., in the solar photosphere and prominences.

Dusty and **self-gravitational** plasmas in space are interesting in view of the diverse and often surprising facts about **planetary rings** and **comet environments**, interstellar dark space.

1.3 Gravitational systems

Gravity plays a central role in the dynamics of many astrophysical systems – from stars to the Universe as a whole.

A **gravitational** force acts on the particles as follows:

$$m_i \, \dot{\mathbf{v}}_i = -m_i \, \nabla \phi \, . \tag{1.41}$$

Here the gravitational potential

$$\phi(t, \mathbf{r}) = -\sum_{n=1}^{N} \frac{G \, m_n}{|\, \mathbf{r}_n(t) - \mathbf{r} \,|} \, , \qquad n \neq i \, , \tag{1.42}$$

G is the gravitational constant.

We shall return to this subject many times, e.g., while studying the **virial theorem**.

This theorem is widely used in astrophysics.

Though the potential (1.42) **looks similar** to the Coulomb potential of charged particles,

> physical properties of gravitational systems differ so much from properties of astrophysical plasma.

We shall see this **fundamental difference** in what follows.

1.4 Practice: Exercises and Answers

Exercise 1.1. Show that

> any distribution function that is a function of the constants of motion – the **invariants** of motion – satisfies Liouville's equation.

Answer.
A general solution of the equations of motion (1.24) depends on $6N$ constants C_i where $i = 1, 2, \dots 6N$.

If the distribution function is a function of these constants of the motion

$$f = f\,(\,C_1,\,\dots C_i,\,\dots C_{6N}\,)\,, \tag{1.43}$$

we rewrite the left-hand side of Equation (1.13) as

$$\frac{Df}{Dt} = \sum_{i=1}^{6N} \left(\frac{DC_i}{Dt} \right) \left(\frac{\partial f}{\partial C_i} \right). \tag{1.44}$$

Because C_i are constants of the motion, $DC_i/Dt = 0$.

Therefore the right-hand side of Equation (1.44) is also zero. Q.e.d.

This is the so-called **Jeans theorem**.

Exercise 1.2. Rewrite the Liouville theorem by using the Hamilton equations.

Answer.

Rewrite the Newton set of equations (1.24) in the Hamilton form:

$$\dot{q}_\alpha = \frac{\partial H}{\partial P_\alpha}, \qquad \dot{P}_\alpha = -\frac{\partial H}{\partial q_\alpha}, \qquad \alpha = 1, 2, 3\,. \tag{1.45}$$

Here $H(P,\, q)$ is the Hamiltonian of a system, q_α and P_α are the **generalized** coordinates and momenta, respectively.

Let us substitute the variables $\mathbf{r}$ and $\mathbf{v}$ in the Liouville equation by the generalized variables $\mathbf{q}$ and $\mathbf{P}$:

$$\frac{\partial f}{\partial t} + \nabla_{\mathbf{P}} H \cdot \nabla_{\mathbf{q}} f - \nabla_{\mathbf{q}} H \cdot \nabla_{\mathbf{P}} f = 0\,. \tag{1.46}$$

Recall that the Poisson brackets for arbitrary quantities A and B are defined to be

$$[\,A\,,\,B\,] = \sum_{\alpha=1}^{3}\left(\frac{\partial A}{\partial q_\alpha}\frac{\partial B}{\partial P_\alpha} - \frac{\partial A}{\partial P_\alpha}\frac{\partial B}{\partial q_\alpha}\right). \tag{1.47}$$

Applying (1.47) to (1.46), we find the final form of the Liouville theorem

$$\boxed{\frac{\partial f}{\partial t} + [\,f\,,\,H\,] = 0\,.} \tag{1.48}$$

Note that for a system in equilibrium

$$[\,f\,,\,H\,] = 0\,. \tag{1.49}$$

Exercise 1.3. Discuss what to do with the Liouville theorem, if it is impossible to disregard **quantum indeterminacy** and assume that the classical description of a system is justified.

Consider the case of dense fluids inside stars, for example, white dwarfs.

Comment.

Inside a **white dwarf** star the temperature $T \sim 10^5$ K, but the density is very high: $n \sim 10^{28} - 10^{30}$ cm^{-3}.

The electrons cannot be regarded as classical particles.

We have to consider them as a quantum system with a Fermi-Dirac distribution.

Exercise 1.4. Recall the Liouville theorem in a course of mechanics – the **phase volume of a system** is independent of t.

Show that this formulation is equivalent to Equation (1.13).

Exercise 1.5. Show that Maxwell's equations imply the **continuity equation for electric charge**.

Answer.

Operating with the divergence on Equation (1.26),

Action:

$$\text{div} \quad ==> \quad \text{rot}\,\mathbf{B} = \frac{4\pi}{c}\mathbf{j} + \frac{1}{c}\frac{\partial\,\mathbf{E}}{\partial t}\,,$$

we have

$$0 = \frac{4\pi}{c}\,\text{div}\,\mathbf{j} + \frac{1}{c}\frac{\partial}{\partial t}\,\text{div}\,\mathbf{E}\,.$$

Substituting (1.29)

Comment:

$$(1.29): \qquad \text{div}\,\mathbf{E} = 4\pi\rho^{\text{q}}\,,$$

in this equation gives us the continuity equation for the electric charge

$$\frac{\partial}{\partial t}\rho^{\text{q}} + \text{div}\,\mathbf{j} = 0\,. \tag{1.50}$$

Exercise 1.6. Starting from Maxwell's equations, derive the **energy conservation law** for an electromagnetic field.

Answer.

Multiply Equation (1.26) by the electric field vector **E** and add it to Equation (1.27) multiplied by the magnetic field vector **B**.

The result is

$$\boxed{\frac{\partial}{\partial t} W = -\,\mathbf{j}\,\mathbf{E} - \operatorname{div}\mathbf{G}\,.} \tag{1.51}$$

Here

$$W = \frac{E^2 + B^2}{8\pi} \tag{1.52}$$

is the **energy of electromagnetic field** in a unit volume of space;

$$\mathbf{G} = \frac{c}{4\pi}\,[\,\mathbf{E} \times \mathbf{B}\,] \tag{1.53}$$

is the flux of electromagnetic field energy through a unit surface in space, i.e. the **Poynting vector**.

The first term on the right-hand side of Equation (1.51) is the power of work done by the electric field on all the charged particles in the unit volume of space.

In the simplest approximation

$$e\,\mathbf{v}\,\mathbf{E} = \frac{d}{dt}\,\mathcal{E}\,, \tag{1.54}$$

where $\mathcal{E}$ is the particle kinetic energy.

Hence instead of Equation (1.51) we write the following form of the **energy conservation law**:

$$\frac{\partial}{\partial t}\left(\frac{E^2 + B^2}{8\pi} + \frac{\rho v^2}{2}\right) + \operatorname{div}\left(\frac{c}{4\pi}\,[\,\mathbf{E} \times \mathbf{B}\,]\right) = 0\,. \quad (1.55)$$

Chapter 2

Statistical Description of Interacting Particle Systems

In a system which consists of many interacting particles, the statistical mechanism of 'mixing' in phase space works and makes the system's behavior *on average* more simple.

2.1 The averaging of Liouville's equation

2.1.1 Averaging over phase space

As was shown above, the **exact** state of a system consisting of N interacting particles can be given by the **exact** distribution function in the 6D phase space $X = \{\,\mathbf{r}, \mathbf{v}\}$.

This function is the **sum** of δ-functions in N points of the phase space:

$$\hat{f}(\mathbf{r}, \mathbf{v}, t) = \sum_{i=1}^{N} \delta\left(\mathbf{r} - \mathbf{r}_i(t)\right)\, \delta\left(\mathbf{v} - \mathbf{v}_i(t)\right). \qquad (2.1)$$

We use Liouville's equation to describe the **change** of the system state:

$$\frac{\partial \hat{f}}{\partial t} + \mathbf{v} \cdot \nabla_{\mathbf{r}} \hat{f} + \frac{\mathbf{F}}{m} \cdot \nabla_{\mathbf{v}} \hat{f} = 0\,. \tag{2.2}$$

Once the exact **initial state** of all the particles is known, it can be represented by N **points** in the phase space (Fig. 2.1).

The motion of these points is described by Liouville's equation.

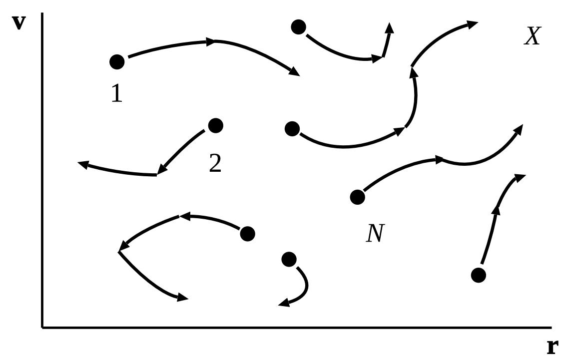


Figure 2.1: Particle trajectories in the 6D phase space X.

In fact we usually know only some **average** characteristics of the system's state, such as the temperature, density, etc.

Moreover the behavior of each single particle is in general of **no** interest.

For this reason, instead of the exact distribution function, let us introduce the distribution function **averaged** over a small volume ΔX of phase space at a moment of time t:

$$\langle \hat{f}(\mathbf{r}, \mathbf{v}, t) \rangle_X = \frac{1}{\Delta X} \int\limits_{\Delta X} \hat{f}(X, t)\, dX \,. \tag{2.3}$$

The **mean** number of particles that present at a moment of time t in an element of volume ΔX is

$$\langle \hat{f}(\mathbf{r}, \mathbf{v}, t) \rangle_X \cdot \Delta X = \int\limits_{\Delta X} \hat{f}(\mathbf{r}, \mathbf{v}, t)\, dX \,. \tag{2.4}$$

Obviously the distribution function averaged over phase volume differs from the exact one (Fig. 2.2).

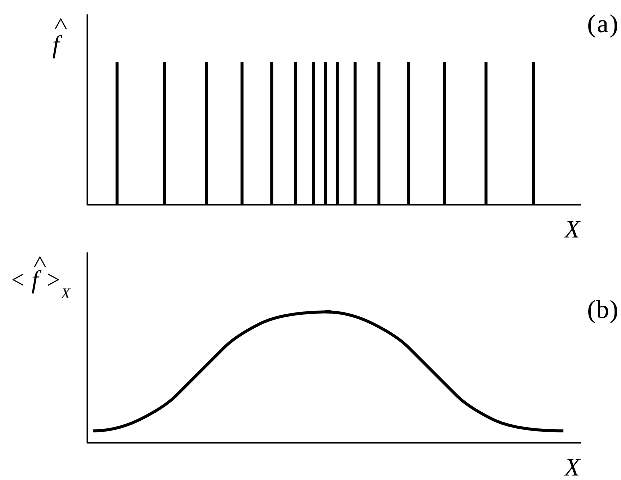


Figure 2.2: The 1D analogy of the distribution function in phase space X: (a) the **exact** distribution function (2.1), (b) the **averaged** function (2.3).

2.1.2 Two statistical postulates

Let us average the exact distribution function (2.1) over a small time interval Δt centered at a moment of time t:

$$\langle \hat{f}(\mathbf{r}, \mathbf{v}, t) \rangle_t = \frac{1}{\Delta t} \int\limits_{\Delta t} \hat{f}(\mathbf{r}, \mathbf{v}, t) \, dt \,. \tag{2.5}$$

Here Δt is small in comparison with the characteristic time of the system's evolution:

$$\Delta t \ll \tau_{ev} \,. \tag{2.6}$$

We assume that the following **two statistical postulates** are applicable to the system considered.

The first postulate:

> The mean values $\langle \hat{f} \rangle_X$ and $\langle \hat{f} \rangle_t$ **exist** for sufficiently small ΔX and Δt and are **independent** of the averaging scales ΔX and Δt.

Clearly the first postulate implies that the number of particles should be **large**.

For a small number of particles the mean value depends upon the averaging scale:

if, e.g., $N = 1$ then the exact distribution function (2.1) is simply a δ-function, and the average over the variable X is

$$\langle \hat{f} \rangle_X = 1/\Delta X \,.$$

For illustration, the case $(\Delta X)_1 > \Delta X$ is shown in Fig. 2.3.

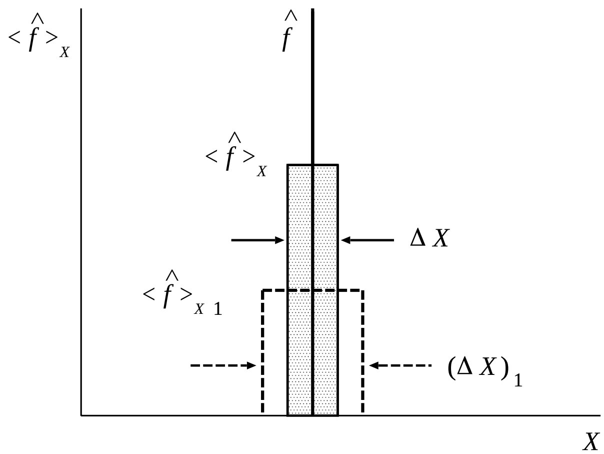


Figure 2.3: Averaging of the exact distribution function $\hat{f}$ which is equal to a δ-function.

The second postulate is

$$\langle \hat{f}(X,t) \rangle_X = \langle \hat{f}(X,t) \rangle_t = f(X,t)\,. \qquad (2.7)$$

The averaging of the distribution function over phase space is **equivalent** to the averaging over time.

While speaking of the small ΔX and Δt, we assume that they are **not too small**:

ΔX must contain a reasonably **large** number of particles while

Δt must be **large** in comparison with the duration of **drastic changes** of the exact distribution function, such as the duration of the particle **collisions**:

$$\Delta t \gg \tau_c \,. \tag{2.8}$$

It is in this case that the statistical mechanism of particle '**mixing**' in phase space is at work and

> the averaging of the exact distribution function over the time Δt is **equivalent** to the averaging over the phase volume ΔX.

2.1.3 A statistical mechanism of mixing

Let us try to understand **qualitatively** how the mixing mechanism works in phase space.

We start from the **dynamical** description of the N-particle system in $6N$-**dimensional** phase space in which

$$\Gamma = \{ \mathbf{r}_i, \, \mathbf{v}_i \} \,, \qquad i = 1, \, 2, \ldots \, N, \tag{2.9}$$

a point is determined ($t = 0$ in Fig. 2.4) by the initial conditions of all the particles.

The motion of this point is described by Liouville's equation.

The point moves along a complicated **dynamical trajectory** because the interactions in a many-particle system are extremely intricate and **complicated**.

The dynamical trajectory has a **remarkable property**.

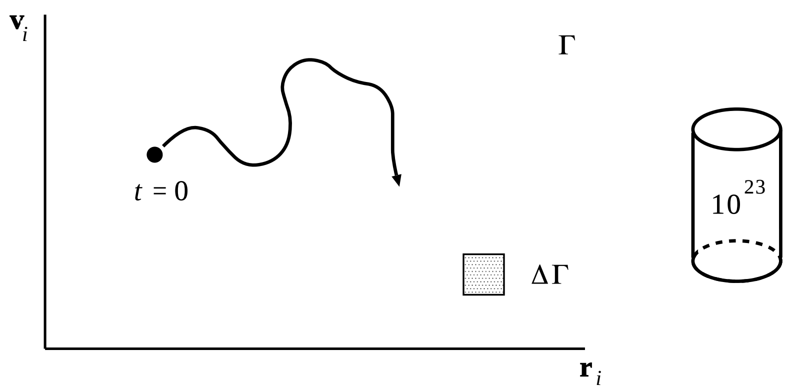


Figure 2.4: The dynamical trajectory of a system of N particles in the $6N$-D phase space Γ.

Imagine a **glass vessel** containing a gas consisting of a large number N of particles.

The state of this gas at any moment of time is depicted by a **single** point in the phase space Γ.

Let us imagine **another** vessel which is identical to the first one, with one exception.

At **any** moment of time t, the gas state in the second vessel is **different** from that in the first one.

These states are depicted by two **different points** in the space Γ.

For example, at $t = 0$, they are points 1 and 2 in Fig. 2.5.

With the passage of time, the gas states in both vessels change, whereas the two points in the space Γ draw two **different** dynamical trajectories (Fig. 2.5).

These trajectories do **not** intersect.

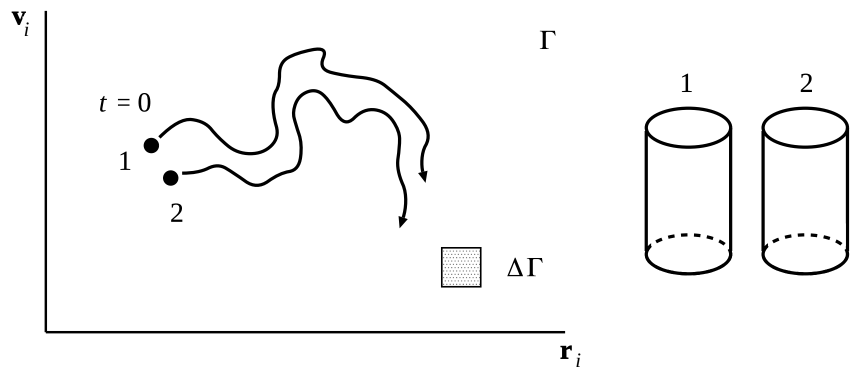


Figure 2.5: The dynamical trajectories of two systems never cross each other.

If they had intersected at just one point, then the state of the first gas, determined by $6N$ numbers $(\mathbf{r}_i, \mathbf{v}_i)$, would have coincided with the state of the second gas.

These numbers could be taken as the **initial conditions** which, in turn, would have uniquely determined the motion.

The two trajectories **would have merged** into one.

For the same reason the trajectory of a system cannot intersect itself.

Thus we come to the conclusion that

> **only one** dynamical trajectory of a many particle system passes through **each** point of the phase space Γ.

Since the trajectories differ in initial conditions, we can introduce an infinite **ensemble of systems** (glass vessels) corresponding to the different initial conditions.

In a **finite** time the ensemble of dynamical trajectories will **closely** fill the phase space Γ, **without intersections**.

By **averaging over the ensemble** we can answer the question:

what is the probability that, at a moment of time t, the system will be found in an element $\Delta\Gamma = \Delta \mathbf{r}_i \, \Delta \mathbf{v}_i$ of the phase space Γ:

$$dw = \langle \, \hat{f}(\mathbf{r}_i, \mathbf{v}_i) \, \rangle_{\Gamma} \; d\Gamma. \tag{2.10}$$

Here $\langle \, \hat{f}(\mathbf{r}_i, \mathbf{v}_i) \, \rangle_{\Gamma}$ is a function of all the coordinates and velocities.

It plays the role of the **probability distribution density** in the phase space Γ and is called the statistical distribution function or simply the **distribution function**.

* * *

It is obvious that the same **probability density** can be obtained in another way – through the **averaging over time**.

The dynamical trajectory of a system, given a sufficient large time Δt, will **closely cover** the space Γ.

Since the trajectory is very intricate, it will **repeatedly pass** through the phase space element $\Delta\Gamma$.

Let $(\Delta t)_{\Gamma}$ be the time during which the system locates in $\Delta\Gamma$.

For a sufficiently large Δt, which is formally restricted by the characteristic time of evolution of the system as a whole, the ratio $(\Delta t)_{\Gamma}/\Delta t$ tends to the limit

$$\lim_{\Delta t \to \infty} \frac{(\,\Delta t\,)_{\Gamma}}{\Delta t} = \frac{dw}{d\,\Gamma} = \langle\, \hat{f}(\mathbf{r}_i, \mathbf{v}_i, t)\,\rangle_{\,t}\,. \tag{2.11}$$

By virtue of the role of the **probability density**, it is clear that

> the statistical averaging over the ensemble (2.10) is equivalent to the averaging over time (2.11) as well as to the definition (2.5).

2.1.4 Derivation of a general kinetic equation

Now we have everything what we need to average the exact Liouville equation

$$\frac{\partial \hat{f}}{\partial t} + \mathbf{v} \cdot \nabla_{\mathbf{r}}\, \hat{f} + \frac{\mathbf{F}}{m} \cdot \nabla_{\mathbf{v}}\, \hat{f} = 0\,.$$

Since the equation contains the derivatives with respect to time t and phase-space coordinates $(\mathbf{r}, \mathbf{v})$, the procedure of averaging is defined as follows:

$$f(X,t) = \frac{1}{\Delta X\,\Delta t} \int\limits_{\Delta X} \int\limits_{\Delta t} \hat{f}(X,t)\,dX\,dt\,. \tag{2.12}$$

Averaging the first term of the Liouville equation gives

$$\frac{1}{\Delta X\,\Delta t}\int\limits_{\Delta X}\int\limits_{\Delta t}\frac{\partial \hat{f}}{\partial t}\,dX\,dt = \frac{1}{\Delta t}\int\limits_{\Delta t}\frac{\partial}{\partial t}\left[\frac{1}{\Delta X}\int\limits_{\Delta X}\hat{f}\,dX\right]dt =$$

$$= \frac{1}{\Delta t}\int\limits_{\Delta t}\frac{\partial}{\partial t}f\,dt = \frac{\partial f}{\partial t}\,. \qquad (2.13)$$

In the last equality the use is made of the fact that, by virtue of the second postulate, the averaging of a smooth averaged function does not change it.

Let us average the second term in Equation (2.2):

$$\frac{1}{\Delta X\,\Delta t}\int\limits_{\Delta X}\int\limits_{\Delta t} v_\alpha\frac{\partial \hat{f}}{\partial r_\alpha}\,dX\,dt =$$

$$= \frac{1}{\Delta X}\int\limits_{\Delta X} v_\alpha\frac{\partial}{\partial r_\alpha}\left[\frac{1}{\Delta t}\int\limits_{\Delta t}\hat{f}\,dt\right]dX =$$

$$= \frac{1}{\Delta X}\int\limits_{\Delta X} v_\alpha\frac{\partial}{\partial r_\alpha}f\,dX = v_\alpha\frac{\partial f}{\partial r_\alpha}\,. \qquad (2.14)$$

Here the index $\alpha = 1, 2, 3$.

To average the term containing the force $\mathbf{F}$, let us represent it as a sum of a **mean force** $\langle\,\mathbf{F}\,\rangle$ and the force due to the difference of the real force field from the mean (smooth) one:

$$\mathbf{F} = \langle\,\mathbf{F}\,\rangle + \mathbf{F}'. \qquad (2.15)$$

Substituting (2.15) in the third term in Equation (2.2) and averaging it, we have

$$\frac{1}{\Delta X\,\Delta t}\int\limits_{\Delta X}\int\limits_{\Delta t}\frac{F_\alpha}{m}\frac{\partial\hat{f}}{\partial v_\alpha}\,dX\,dt =$$

$$= \frac{\langle\, F_\alpha\,\rangle}{m}\,\frac{1}{\Delta X}\int\limits_{\Delta X}\frac{\partial}{\partial v_\alpha}\left[\frac{1}{\Delta t}\int\limits_{\Delta t}\hat{f}\,dt\right]dX +$$

$$+\frac{1}{\Delta X\,\Delta t}\int\limits_{\Delta X}\int\limits_{\Delta t}\frac{F'_\alpha}{m}\frac{\partial\hat{f}}{\partial v_\alpha}\,dX\,dt =$$

$$= \frac{\langle\, F_\alpha\,\rangle}{m}\frac{\partial f}{\partial v_\alpha} + \frac{1}{\Delta X\,\Delta t}\int\limits_{\Delta X}\int\limits_{\Delta t}\frac{F'_\alpha}{m}\frac{\partial\hat{f}}{\partial v_\alpha}\,dX\,dt\,. \tag{2.16}$$

Gathering all three terms together, we write the averaged Liouville equation in the form

$$\boxed{\frac{\partial f}{\partial t} + \mathbf{v}\cdot\nabla_{\mathbf{r}}\,f + \frac{\langle\,\mathbf{F}\,\rangle}{m}\cdot\nabla_{\mathbf{v}}\,f = \left(\frac{\partial\hat{f}}{\partial t}\right)_{\mathrm{c}}\,,} \tag{2.17}$$

where

$$\boxed{\left(\frac{\partial\hat{f}}{\partial t}\right)_{\mathrm{c}} = -\,\frac{1}{\Delta X\,\Delta t}\int\limits_{\Delta X}\int\limits_{\Delta t}\frac{F'_\alpha}{m}\frac{\partial\hat{f}}{\partial v_\alpha}\,dX\,dt\ .} \tag{2.18}$$

Equation (2.17) and its right-hand side (2.18) are called the **kinetic equation** and the **collisional integral**, respectively.

Thus we have found the **most general** form of the kinetic equation with a collisional integral, which cannot be directly used in plasma astrophysics, without making some **additional simplifying assumptions**.

The main of them is the **binary character** of collisions.

2.2 A collisional integral and correlation functions

2.2.1 Binary interactions

The statistical mechanism of mixing in phase space makes particles have **no individuality**.

However, we have to distinguish **different kinds** of particles, e.g., electrons and protons, because their **behaviors differ**.

Let $\hat{f}_k\,(\mathbf{r}, \mathbf{v}, t)$ be the exact distribution function of particles of the **kind** k

$$\hat{f}_k\,(\mathbf{r}, \mathbf{v}, t) = \sum_{i=1}^{N_k} \delta\,(\mathbf{r} - \mathbf{r}_{ki}(t))\;\delta\,(\mathbf{v} - \mathbf{v}_{ki}(t))\,, \qquad (2.19)$$

the index i denoting the ith particle of kind k, N_k being the number of particles of kind k.

The Liouville equation for the particles of kind k takes a view

$$\frac{\partial \hat{f}_k}{\partial t} + \mathbf{v} \cdot \nabla_{\mathbf{r}} \hat{f}_k + \frac{\hat{\mathbf{F}}_k}{m_k} \cdot \nabla_{\mathbf{v}} \hat{f}_k = 0 \, , \tag{2.20}$$

m_k is the mass of a particle of kind k.

The force acting on a particle of kind k at a point $(\mathbf{r}, \mathbf{v})$ of the phase space X at a moment of time t, $\hat{F}_{k,\alpha}(\mathbf{r}, \mathbf{v}, t)$, is the sum of forces acting on this particle from all other particles (Fig. 2.6):

$$\hat{F}_{k,\alpha}(\mathbf{r}, \mathbf{v}, t) = \sum_l \sum_{i=1}^{N_l} \hat{F}^{(i)}_{kl,\alpha}(\mathbf{r}, \mathbf{v}, \mathbf{r}_{li}(t), \mathbf{v}_{li}(t)) \; . \tag{2.21}$$

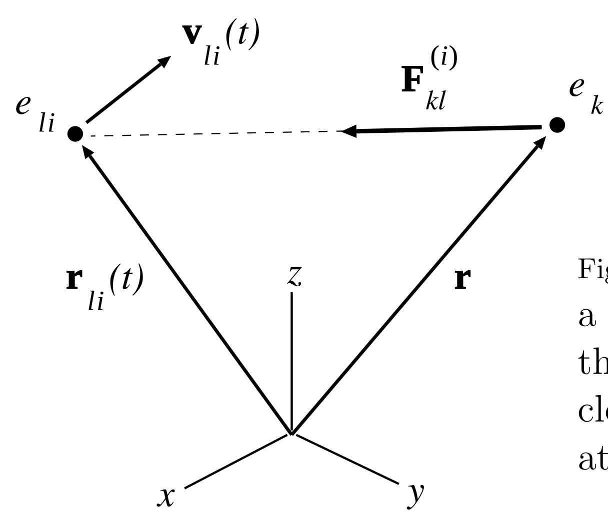


Figure 2.6: An action of a particle e_{li} located at the point $\mathbf{r}_{li}$ on a particle of kind k at a point $\mathbf{r}$ at a moment of time t.

So the **total force** $\hat{F}_{k,\alpha}(\mathbf{r}, \mathbf{v}, t)$ depends upon the instant positions and velocities of **all** the particles.

By using the exact distribution function, we rewrite formula (2.21) as follows:

$$\hat{F}_{k,\alpha}(\mathbf{r}, \mathbf{v}, t) = \sum_{l} \int_{X_1} F_{kl,\alpha}(X, X_1)\ \hat{f}_l(X_1, t)\ dX_1\,. \quad (2.22)$$

Here

we assume that an **interaction law** $F_{kl,\alpha}(X, X_1)$ is explicitly independent of time t;

$\hat{f}_l(X, t)$ is the exact distribution function of particles of kind l,

the variable of integration is designated as $X_1 = \{\, \mathbf{r}_1, \mathbf{v}_1 \,\}$ and $dX_1 = d^3\mathbf{r}_1\, d^3\mathbf{v}_1$.

> Formula (2.22) takes into account that the forces considered are **binary** ones, i.e. they can be represented as a sum of interactions between **two** particles.

Making use of the representation (2.22), let us average the **force term** in the Liouville equation, as this has been done in formula (2.16).

We have

$$\frac{1}{\Delta X\, \Delta t} \int_{\Delta X} \int_{\Delta t} \frac{1}{m_k}\, \hat{F}_{k,\alpha}(\mathbf{r}, \mathbf{v}, t)\ \frac{\partial \hat{f}_k}{\partial v_\alpha}\, dX\, dt =$$

$$= \frac{1}{\Delta X \, \Delta t} \int\limits_{\Delta X} \int\limits_{\Delta t} \sum_{l} \int\limits_{X_1} \frac{1}{m_k} \, F_{kl,\alpha} \left(X, X_1 \right) \, \hat{f}_l \left(X_1, t \right) \times$$

$$\times \frac{\partial}{\partial v_\alpha} \, \hat{f}_k \left(X, t \right) \, dX \, \, dX_1 \, dt =$$

$$= \frac{1}{\Delta X} \int\limits_{\Delta X} \sum_{l} \int\limits_{X_1} \frac{1}{m_k} \, F_{kl,\alpha} \left(X, X_1 \right) \, \times$$

$$\times \frac{\partial}{\partial v_\alpha} \left[\frac{1}{\Delta t} \int\limits_{\Delta t} \, \hat{f}_k \left(X, t \right) \, \hat{f}_l \left(X_1, t \right) \, dt \right] dX \, dX_1 \, . \qquad (2.23)$$

Here we have taken into account that the exact distribution function $\hat{f}_l \left(X_1, t \right)$ is independent of the velocity $\mathbf{v}$, which is a part of the variable $X = \{ \, \mathbf{r}, \, \mathbf{v} \, \}$ related to the particles of the kind k.

> Formula (2.23) contains the **pair products** of exact distribution functions of different particle kinds, as is natural for the case of **binary interactions**.

2.2.2 Binary correlation

Let us represent the exact distribution function $\hat{f}_k$ as

$$\hat{f}_k \left(X, t \right) = f_k \left(X, t \right) + \hat{\varphi}_k \left(X, t \right) , \qquad (2.24)$$

where

$f_k(X, t)$ is the **statistically averaged** distribution function,

$\hat{\varphi}_k(X, t)$ is the deviation of the exact distribution function from the averaged one.

It is obvious that, according to (2.24),

$$\hat{\varphi}_k(X, t) = \hat{f}_k(X, t) - f_k(X, t);$$

hence

$$\langle\, \hat{\varphi}_k(X, t)\, \rangle = 0\,. \tag{2.25}$$

Let us consider the **integrals of pair products** in the **averaged force** term (2.23).

In view of definition (2.24), they can be rewritten as

$$\frac{1}{\Delta t}\int\limits_{\Delta t} \hat{f}_k(X, t)\, \hat{f}_l(X_1, t)\, dt =$$

$$= f_k(X, t)\, f_l(X_1, t) + f_{kl}(X, X_1, t)\,, \tag{2.26}$$

where

$$f_{kl}(X, X_1, t) = \frac{1}{\Delta t}\int\limits_{\Delta t} \hat{\varphi}_k(X, t)\, \hat{\varphi}_l(X_1, t)\, dt\,. \tag{2.27}$$

The function f_{kl} is referred to as the **correlation function** or, more exactly, the **binary** correlation function.

The **physical meaning** of the correlation function is clear from (2.26).

The left-hand side of (2.26) means the probability to find a particle of kind k at a point X of the phase space at a moment of time t **under condition** that a particle of kind l places at a point X_1 at the same time.

By definition this is a **conditional probability**.

In the right-hand side of (2.26) the distribution function $f_k\,(X,t)$ characterizes the probability that a particle of kind k stays at a point X at a moment of time t.

The function $f_l\,(X_1,t)$ plays the analogous role for the particles of kind l.

> If the particles of kind k did **not** interact with those of kind l, then their distributions would be independent, i.e. probability densities would **simply multiply**:

$$\langle\, \hat{f}_k\,(X,t)\;\hat{f}_l\,(X_1,t)\,\rangle = f_k\,(X,t)\;f_l\,(X_1,t)\,. \tag{2.28}$$

So in the right-hand side of (2.26) there should be

$$f_{kl}\,(X,X_1,t) = 0\,. \tag{2.29}$$

There would be **no correlation** in the particle distribution.

We consider a system of interacting particles.

With the proviso that the parameter characterizing the binary interaction, e.g., Coulomb collision considered below,

$$\zeta_{\rm i} \approx \frac{e^2}{\langle\, l \,\rangle} \Big/ \left\langle \frac{mv^2}{2} \right\rangle , \tag{2.30}$$

is small under conditions in a wide range, the correlation function must be **relatively small**.

> If the interaction is **weak**, the second term in the right-hand side of (2.26) must be **small** in comparison with the first one.

This fundamental property allows us to construct a theory of plasma in many cases of astrophysical interest.

2.2.3 The collisional integral and binary correlation

Now let us substitute (2.26) in formula (2.23) for the **averaged force** term:

$$\frac{1}{\Delta X\, \Delta t} \int\limits_{\Delta X} \int\limits_{\Delta t} \frac{1}{m_k}\, \hat{F}_{k,\alpha}\,(X,t)\, \frac{\partial \hat{f}_k}{\partial v_\alpha}\, dX\, dt =$$

$$= \frac{1}{\Delta X} \int\limits_{\Delta X} \sum_l \int\limits_{X_1} \frac{1}{m_k}\, F_{kl,\alpha}\,(X, X_1)\, \frac{\partial}{\partial v_\alpha}\, [\, f_k\,(X,t)\;\, f_l\,(X_1,t) \, +$$

$$+ \, f_{kl}\,(X, X_1, t)\,]\; dX\, dX_1 =$$

since $f_k\,(X,t)$ is a smooth function, its derivative over v_α can be brought out of the averaging procedure:

$$= \left[\frac{\partial}{\partial v_\alpha} f_k (X,t) \right] \times$$

$$\times \left\{ \frac{1}{\Delta X} \int\limits_{\Delta X} \sum_l \int\limits_{X_1} \frac{1}{m_k} F_{kl,\alpha} (X, X_1) \ f_l (X_1, t) \, dX \, dX_1 \right\} +$$

$$+ \frac{1}{\Delta X} \int\limits_{\Delta X} \sum_l \int\limits_{X_1} \frac{1}{m_k} F_{kl,\alpha} (X, X_1) \ \frac{\partial}{\partial v_\alpha} f_{kl} (X, X_1, t) \, dX \, dX_1 =$$

$$= \frac{1}{m_k} F_{k,\alpha} (X,t) \ \frac{\partial f_k (X,t)}{\partial v_\alpha} +$$

$$+ \sum_l \int\limits_{X_1} \frac{1}{m_k} F_{kl,\alpha} (X, X_1) \ \frac{\partial f_{kl} \ (X, X_1, t)}{\partial v_\alpha} \ dX_1 \, . \qquad (2.31)$$

Here we have taken into account that the averaging of smooth functions does not change them, and the following definition of the **averaged force** is used:

$$F_{k,\alpha} (X,t) = \frac{1}{\Delta X} \int\limits_{\Delta X} \sum_l \int\limits_{X_1} F_{kl,\alpha} (X, X_1) \ \ f_l (X_1, t) \ dX \, dX_1 =$$

$$= \sum_l \int\limits_{X_1} F_{kl,\alpha} (X, X_1) \ f_l (X_1, t) \, dX_1 \, . \qquad (2.32)$$

This definition coincides with the previous definition (2.16) of the averaged force, since all the deviations of the real force $\hat{\mathbf{F}}_k$ from the mean (smooth) force $\mathbf{F}_k$ are taken care of in the deviations $\hat{\varphi}_k$ and $\hat{\varphi}_l$ of the real distribution functions $\hat{f}_k$ and $\hat{f}_l$ from their mean values f_k and f_l.

Thus the **collisional integral** is represented in the form

$$\left(\frac{\partial \hat{f}_k}{\partial t}\right)_{\mathrm{c}} = -\sum_l \int\limits_{X_1} \frac{1}{m_k} F_{kl,\alpha}(X, X_1) \, \frac{\partial f_{kl}(X, X_1, t)}{\partial v_\alpha} \, dX_1 \, . \tag{2.33}$$

Let us recall that for the **Lorentz force** as well as for the **gravitational** one the condition

$$\frac{\partial}{\partial v_\alpha} F_{kl,\alpha}(X, X_1) = 0 \tag{2.34}$$

is satisfied.

So, we obtain from formula (2.33) the following expression

$$\left(\frac{\partial \hat{f}_k}{\partial t}\right)_{\mathrm{c}} = -\frac{\partial}{\partial v_\alpha} \sum_l \int\limits_{X_1} \frac{1}{m_k} F_{kl,\alpha}(X, X_1) \, f_{kl}(X, X_1, t) \, dX_1 \, . \tag{2.35}$$

Hence the collisional integral can be written in the **divergent form** in the **velocity space v** :

$$\boxed{\left(\frac{\partial \hat{f}_k}{\partial t} \right)_c = - \frac{\partial}{\partial v_\alpha} J_{k,\alpha} \,,} \tag{2.36}$$

where the **flux of particles** of kind k in the **velocity space** is

$$J_{k,\alpha}(X,t) = \sum_l \int_{X_1} \frac{1}{m_k} F_{kl,\alpha}(X,X_1) \; f_{kl}(X,X_1,t)\, dX_1 \,. \tag{2.37}$$

Therefore the averaged Liouville equation or **the kinetic equation for particles of kind** k

$$\frac{\partial f_k(X,t)}{\partial t} + v_\alpha \frac{\partial f_k(X,t)}{\partial r_\alpha} + \frac{F_{k,\alpha}(X,t)}{m_k} \frac{\partial f_k(X,t)}{\partial v_\alpha} =$$

$$= - \frac{\partial}{\partial v_\alpha} \sum_l \int_{X_1} \frac{1}{m_k} F_{kl,\alpha}(X,X_1) \; f_{kl}(X,X_1,t)\, dX_1 \tag{2.38}$$

contains the **unknown** function f_{kl}.

Hence the kinetic Equation (2.38) for distribution function f_k is **not closed**.

We have to find the **equation for the correlation function** f_{kl} .

2.3 Equations for correlation functions

To derive the equations for correlation functions, it is **not** necessary to introduce any new postulates or develop new formalisms.

All the necessary equations and averaging procedures are at hand.

Looking at definition

$$f_{kl}\,(X, X_1, t) = \frac{1}{\Delta t} \int\limits_{\Delta t} \hat{\varphi}_k\,(X, t)\,\hat{\varphi}_l\,(X_1, t)\,dt\,,$$

where

$$\hat{\varphi}_k\,(X, t) = \hat{f}_k\,(X, t) - f_k\,(X, t)\,,$$

we see that we need an equation which will describe the deviation of distribution function from its mean value, i.e. the function $\hat{\varphi}_k = \hat{f}_k - f_k$.

In order to derive such equation, we simply have to subtract the **averaged** Liouville equation

$$\frac{\partial f_k\,(X, t)}{\partial t} \;+\; v_\alpha\,\frac{\partial f_k\,(X, t)}{\partial r_\alpha} \;+\; ... \;=\; ...$$

from the **exact** Liouville equation (2.2)

$$\frac{\partial \hat{f}_k}{\partial t} + \mathbf{v} \cdot \nabla_{\mathbf{r}}\,\hat{f}_k + \frac{\hat{\mathbf{F}}_k}{m_k} \cdot \nabla_{\mathbf{v}}\,\hat{f}_k = 0\,.$$

The result is

$$\frac{\partial\,\hat{\varphi}_k\,(X,t)}{\partial t} + v_\alpha\,\frac{\partial\,\hat{\varphi}_k\,(X,t)}{\partial r_\alpha} + \frac{\hat{F}_{k,\alpha}}{m_k}\,\frac{\partial\,\hat{f}_k}{\partial v_\alpha} - \frac{F_{k,\alpha}}{m_k}\,\frac{\partial\,f_k}{\partial v_\alpha} =$$

$$= \frac{\partial}{\partial v_\alpha}\sum_l \int\limits_{X_1} \frac{1}{m_k}\,F_{kl,\alpha}\,(X, X_1)\;f_{kl}\,(X, X_1)\;dX_1\,. \qquad (2.39)$$

Here

$$\hat{F}_{k,\alpha}\,(X,t) = \sum_l \int\limits_{X_1} F_{kl,\alpha}\,(X, X_1)\,\hat{f}_l\,(X_1,t)\;dX_1 \qquad (2.40)$$

is the **exact** force (2.22) acting on a particle of the kind k, and

$$F_{k,\alpha}\,(X,t) = \sum_l \int\limits_{X_1} F_{kl,\alpha}\,(X, X_1)\,f_l\,(X_1,t)\;dX_1 \qquad (2.41)$$

is the statistically **averaged** force.

Considering that we need the equation for the pair correlation function

$$f_{kl}\,(X_1, X_2, t) = \langle\; \hat{\varphi}_k\,(X_1,t)\;\hat{\varphi}_l\,(X_2,t)\;\rangle\,,$$

let us take **two equations**:

one for $\hat{\varphi}_k\,(X_1,t)$

$$\frac{\partial\,\hat{\varphi}_k\,(X_1,t)}{\partial t} + v_{1,\alpha}\,\frac{\partial\,\hat{\varphi}_k\,(X_1,t)}{\partial\,r_{1,\alpha}} + \ldots = 0 \qquad (2.42)$$

and another for $\hat{\varphi}_l\,(X_2, t)$

$$\frac{\partial\,\hat{\varphi}_l\,(X_2, t)}{\partial t} + v_{2,\alpha}\,\frac{\partial\,\hat{\varphi}_l\,(X_2, t)}{\partial\,r_{2,\alpha}} + \ldots = 0\,. \qquad (2.43)$$

Now we add the equations resulting from (2.42) multiplied by $\hat{\varphi}_l$ and (2.43) multiplied by $\hat{\varphi}_k$.

We obtain

$$\hat{\varphi}_l\,\frac{\partial\,\hat{\varphi}_k}{\partial t} + \hat{\varphi}_k\,\frac{\partial\,\hat{\varphi}_l}{\partial t} + v_{1,\alpha}\,\frac{\partial\,\hat{\varphi}_k}{\partial\,r_{1,\alpha}}\,\hat{\varphi}_l + \ldots = 0$$

or

$$\frac{\partial\,(\hat{\varphi}_k\,\hat{\varphi}_l)}{\partial t} + v_{1,\alpha}\,\frac{\partial\,(\hat{\varphi}_k\,\hat{\varphi}_l)}{\partial\,r_{1,\alpha}} + v_{2,\alpha}\,\frac{\partial\,(\hat{\varphi}_k\,\hat{\varphi}_l)}{\partial\,r_{2,\alpha}} + \ldots = 0\,. \quad (2.44)$$

On averaging Equation (2.44) we have the equation for the **pair correlation** function:

$$\begin{aligned}
&\frac{\partial f_{kl}\,(X_1, X_2, t)}{\partial t} + \\
&\quad + v_{1,\alpha}\,\frac{\partial f_{kl}\,(X_1, X_2, t)}{\partial\,r_{1,\alpha}} + v_{2,\alpha}\,\frac{\partial f_{kl}\,(X_1, X_2, t)}{\partial\,r_{2,\alpha}} + \\
&\quad + \frac{F_{k,\alpha}\,(X_1, t)}{m_k}\,\frac{\partial f_{kl}\,(X_1, X_2, t)}{\partial\,v_{1,\alpha}} + \frac{F_{l,\alpha}\,(X_2, t)}{m_l}\,\frac{\partial f_{kl}\,(X_1, X_2, t)}{\partial\,v_{2,\alpha}} + \\
&\quad + \frac{\partial f_k\,(X_1, t)}{\partial\,v_{1,\alpha}}\,\sum_n \int\limits_{X_3} \frac{1}{m_k}\,F_{kn,\alpha}\,(X_1, X_3)\,f_{nl}\,(X_3, X_2, t)\,dX_3 +
\end{aligned}$$

$$+\frac{\partial f_l(X_2,t)}{\partial v_{2,\alpha}} \sum_n \int\limits_{X_3} \frac{1}{m_l} F_{ln,\alpha}(X_2,X_3)\; f_{nk}(X_3,X_1,t)\; dX_3 =$$
$$= -\frac{\partial}{\partial v_{1,\alpha}} \sum_n \int\limits_{X_3} \frac{1}{m_k} F_{kn,\alpha}(X_1,X_3)\; f_{kln}(X_1,X_2,X_3,t)\; dX_3 -$$
$$-\frac{\partial}{\partial v_{2,\alpha}} \sum_n \int\limits_{X_3} \frac{1}{m_l} F_{ln,\alpha}(X_2,X_3)\; f_{kln}(X_1,X_2,X_3,t)\; dX_3 . \quad (2.45)$$

Here

$$f_{kln}(X_1,X_2,X_3,t) = \frac{1}{\Delta t} \int\limits_{\Delta t} \hat{\varphi}_k(X_1,t)\; \hat{\varphi}_l(X_2,t)\; \hat{\varphi}_n(X_3,t)\; dt \quad (2.46)$$

is the function of **triple correlations**.

Thus Equation (2.45) for the pair correlation function contains the **unknown** function of triple correlations.

In general,

> the chain of equations for correlation functions is **unclosed**: the equation for the correlation function of sth order contains the function of the order $(s+1)$.

2.4 Practice: Exercises and Answers

Exercise 2.1. By analogy with formula (2.26), show that

$$\langle\, \hat{f}_k(X_1,t)\; \hat{f}_l(X_2,t)\; \hat{f}_n(X_3,t)\, \rangle = \quad (2.47)$$

$$= f_k\,(X_1,t)\;f_l\,(X_2,t)\;f_n\,(X_3,t) +$$

$$+ f_k\,(X_1,t)\;f_{ln}\,(X_2,X_3,t) \,+\, f_l\,(X_2,t)\;f_{kn}\,(X_1,X_3,t) +$$

$$+ f_n\,(X_3,t)\;f_{kl}\,(X_1,X_2,t) \,+\, f_{kln}\,(X_1,X_2,X_3,t)\,.$$

Exercise 2.2. Discuss a similarity and difference between the kinetic theory presented in this Chapter and the famous BBGKY hierarchy theory developed by Bogoliubov, Born and Green, Kirkwood, and Yvon.

Hint. Show that essential to both derivations is the weak-coupling assumption, according to which

> grazing encounters, involving **small** fractional energy and momentum **exchange** between colliding particles, dominate the evolution of the velocity distribution function.

The **weak-coupling** assumption provides justification of the widely appreciated practice which leads to a very significant simplification of the original collisional integral.

Chapter 3

Weakly-Coupled Systems with Binary Collisions

> In a system of many interacting particles, the weak-coupling assumption allows us to introduce a **well controlled approximation** to consider the chain of the equations for correlation functions.
>
> This leads to a **significant simplification** of the collisional integral in astrophysical plasma but not in self-gravitating systems.

3.1 Approximations for binary collisions

3.1.1 The small parameter of kinetic theory

The infinite chain of equations for the correlation functions does **not** contain more information in itself than the Liouville equation for the exact distribution function.

Actually, the statistical smoothing allows to lose '**useless information**' – the information about the **exact** motion of particles.

The value of the chain is that it allows a direct introduction of **new** physical assumptions which make it possible **to break the chain off** at some term (Fig. 3.1) and to estimate the resulting error.

We call this procedure a **well controlled approximation** because it looks, in a sense, similar to the Taylor expansion series.

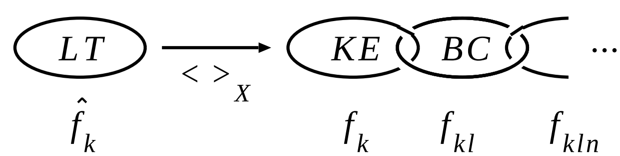


Figure 3.1: How to break the infinite chain of the equations for correlation functions? LT is the Liouville theorem for an exact distribution function $\hat{f}_k$. KE and BC are the kinetic Equation for f_k etc.

There is **no universal way** of breaking the chain off.

It is intimately related, in particular, to the **physical state** of a plasma.

Different states (as well as different **aims**) require different approximations.

The physical state of a plasma can be characterized, at least partially, by the ratio of the mean energy of two particle interaction to their mean kinetic energy

$$\zeta_{\rm i} \approx \frac{e^2}{\langle\, l\,\rangle} \Big/ \left\langle \frac{mv^2}{2} \right\rangle ,$$

If mean kinetic energy can be reasonably characterized by some **effective temperature** T, then

$$\zeta_{\rm i} \approx \frac{e^2}{\langle\, l\,\rangle} \left(k_{\rm B} T \right)^{-1} . \tag{3.1}$$

As a mean distance between the particles we take

$$\langle\, l\,\rangle \approx n^{-1/3} .$$

Hence

$$\zeta_{\rm i} = \frac{e^2}{k_{\rm B}} \times \frac{n^{1/3}}{T} \tag{3.2}$$

is termed the **interaction parameter**.

It is small for a sufficiently **hot** and **rarefied** plasma.

In many astrophysical plasmas, e.g., in the **solar corona**, the interaction parameter is very small.
So

> the thermal kinetic energy of plasma particles is much larger than their interaction energy.

The particles are almost free or moving on definite trajectories in the external fields if the later are present.

We call this case the approximation of **weak** Coulomb interaction.

While constructing a kinetic theory, it is natural to use the **perturbation procedure** with respect to the **small parameter** ζ_{i}.

This means that

> the distribution function f_k must be taken to be of order unity, the pair correlation function f_{kl} of order ζ_{i}, the triple correlation function f_{kln} of order ζ_{i}^2, etc.

We shall see in what follows that this principle has a deep physical sense in kinetic theory.

Such plasmas are said to be '**weakly coupled**'.

An opposite case, when the interaction parameter takes values larger than unity, is **dense**, relatively **cold** plasmas, for example in the interiors of white dwarf stars.

These plasmas are '**strongly coupled**'.

3.1.2 The Vlasov kinetic equation

In the zeroth order with respect to the small parameter ζ_{i}, we obtain the Vlasov equation **with the self-consistent electromagnetic field**:

$$\frac{\partial f_k(X,t)}{\partial t} + v_\alpha \frac{\partial f_k(X,t)}{\partial r_\alpha} +$$

$$+\frac{e_k}{m_k}\left(\mathbf{E}+\frac{1}{c}\,\mathbf{v}\times\mathbf{B}\right)_\alpha \frac{\partial f_k\,(X,t)}{\partial v_\alpha}=0\,. \qquad (3.3)$$

Here **E** and **B** are the **statistically averaged** electric and magnetic fields obeying Maxwell's equations:

$$\begin{aligned} &\operatorname{curl}\mathbf{E}=-\frac{1}{c}\,\frac{\partial\,\mathbf{B}}{\partial t}\,, \qquad \operatorname{div}\mathbf{E}=4\pi\,(\,\rho^{\,0}+\rho^{\,\mathrm{q}}\,)\,,\\ &\operatorname{curl}\mathbf{B}=\frac{1}{c}\,\frac{\partial\,\mathbf{E}}{\partial t}+\frac{4\pi}{c}\,(\,\mathbf{j}^{\,0}+\mathbf{j}^{\,\mathrm{q}}\,)\,, \qquad \operatorname{div}\mathbf{B}=0\,. \end{aligned} \qquad (3.4)$$

$\rho^{\,0}$ and $\mathbf{j}^{\,0}$ are the external charges and currents; they describe the **external** fields, e.g., the uniform magnetic field $\mathbf{B}_0$.

$\rho^{\,\mathrm{q}}$ and $\mathbf{j}^{\,\mathrm{q}}$ are the **statistically smoothed** charge and current due to the plasma particles:

$$\rho^{\,\mathrm{q}}\,(\mathbf{r},t)=\sum_k\,e_k\int_{\mathbf{v}} f_k\,(\mathbf{r},\mathbf{v},t)\,d^{\,3}\mathbf{v}\,, \qquad (3.5)$$

$$\mathbf{j}^{\,\mathrm{q}}\,(\mathbf{r},t)=\sum_k\,e_k\int_{\mathbf{v}} \mathbf{v}\,f_k\,(\mathbf{r},\mathbf{v},t)\,d^{\,3}\mathbf{v}\,. \qquad (3.6)$$

Therefore the electric and magnetic fields are also **statistically smoothed**.

If we are considering processes which occur on a time scale much shorter than the time of collisions,

$$\tau_{ev}\ll\tau_c\,, \qquad (3.7)$$

we use a description which includes the **averaged** electric and magnetic fields but **neglects the microfields responsible for binary collisions**.

This means that

$$\mathbf{F}' = 0\,,$$

therefore the collisional integral is also equal to zero.

The Vlasov equation together with the definitions (3.5) and (3.6), and with Maxwell's Equations (3.4) is a **nonlinear integro-differential** equation.

It serves as a classic basis for the theory of **oscillations and waves** in a plasma with the small parameter $\zeta_{\rm i}$.

The Vlasov equation is also a proper basis for theory of **wave-particle interactions** in astrophysical plasma and collisionless **shock waves**, collisionless **reconnecting current layers**.

3.1.3 The Landau collisional integral

Using the perturbation procedure with respect to the small parameter $\zeta_{\rm i}$ **in the first order**, and neglecting the **close** Coulomb collisions, we find the kinetic equation with the collisional integral given by Landau

$$\left(\frac{\partial \hat{f}_k}{\partial t} \right)_{\rm c} = - \frac{\partial}{\partial v_\alpha}\, J_{k,\alpha}\,, \tag{3.8}$$

Here the flux of particles of kind k in the velocity space is

$$J_{k,\alpha} = \frac{\pi e_k^2 \ln \Lambda}{m_k} \sum_l e_l^2 \int_{\mathbf{v}_l} \left\{ f_k \frac{\partial f_l}{m_l \, \partial v_{l,\beta}} - f_l \frac{\partial f_k}{m_k \, \partial v_{k,\beta}} \right\} \times$$

$$\times \frac{(u^2 \, \delta_{\alpha\beta} - u_\alpha u_\beta)}{u^3} \; d^3 \mathbf{v}_l \, . \tag{3.9}$$

$\mathbf{u} = \mathbf{v} - \mathbf{v}_l$ is the relative velocity, $d^3 \mathbf{v}_l$ corresponds to the integration over the whole velocity space of '**field**' particles l.

$\ln \Lambda$ is the Coulomb logarithm which takes into account divergence of the Coulomb-collision cross-section.

The kinetic equation with the Landau integral is a **non-linear integro-differential equation** for the distribution function $f_k(\mathbf{r}, \mathbf{v}, t)$.

Two approaches correspond to **different limiting cases**.

> The Landau integral takes into account the part of the particle interaction which determines **dissipation** while the Vlasov equation allows for the averaged field, and is thus **reversible**.

For example, in the Vlasov theory, the question of the role of collisions in the neighbourhood of **resonances** remains open.

The famous paper by Landau (1946) was devoted to this problem.

Landau used the reversible Vlasov equation as the basis to study the dynamics of a small perturbation of the Maxwell distribution function, $f^{(1)}(\mathbf{r}, \mathbf{v}, t)$.

In order to solve the linearized Vlasov equation, he made use of the Laplace transformation, and defined the rule to avoid a pole at

$$\omega = k_{\parallel} v_{\parallel}$$

in the divergent integral by the replacement

$$\omega \to \omega + \mathrm{i}\,0\,.$$

This technique for avoiding singularities may be formally replaced by a different procedure.

Namely it is possible to add a **small dissipative term** $-\nu f^{(1)}(\mathbf{r}, \mathbf{v}, t)$ to the right-hand side of the linearized Vlasov equation.

In this way, the Fourier transform of the kinetic equation involves the complex frequency

$$\omega = \omega' + \mathrm{i}\,\nu\,,$$

leading with $\nu \to 0$ to the **same** expression for the **Landau damping**.

Note, however, that

> the Landau damping is **not** by collisions but by a transfer of wave field energy into oscillations of resonant particles.

The Landau method is really a beautiful example of complex analysis leading to an important **new** physical result.

The second approach reduces the reversible Vlasov equation to an **irreversible** one.

Although the dissipation is assumed to be negligibly small, one cannot take the limit $\nu \to 0$ directly in the master equations: this can be done only in the final formulae.

This method of introducing a **collisional damping** is natural.

It shows that

> even very rare collisions play the principal role in the physics of collisionless plasma.

3.1.4 The Fokker-Planck equation

The smallness of the interaction parameter signifies that, in the collisional integral, the sufficiently **distant** Coulomb collisions are taken care of as the interactions with a **small momentum and energy transfer**.

For this reason, it comes as **no** surprise that the Landau integral can be considered as a particular case of a different approach which is the Fokker-Planck equation.

Let us consider a distribution function **independent of space** so that $f = f(\mathbf{v}, t)$.

The Fokker-Planck equation describes the distribution function evolution due to **nonstop overlapping weak collisions** resulting in particle diffusion in velocity space:

$$\frac{\partial f}{\partial t} = \left(\frac{\partial \hat{f}}{\partial t} \right)_{\text{c}} = - \frac{\partial}{\partial v_\alpha} \left[a_\alpha f \right] + \frac{\partial^2}{\partial v_\alpha \, \partial v_\beta} \left[b_{\alpha\beta} \, f \right] . \quad (3.10)$$

This equation coincides with the diffusion equation for some admixture with concentration f, e.g., **Brownian particles** in a gas, on which **stochastic** forces are exerted by the molecules of the gas.

The coefficient $b_{\alpha\beta}$ plays the role of the **diffusion** coefficient and is expressed in terms of the averaged velocity change $\langle \, \delta v_\alpha \, \rangle$ in an elementary act – a collision:

$$b_{\alpha\beta} = \frac{1}{2} \, \langle \, \delta v_\alpha \, \delta v_\beta \, \rangle \, . \quad (3.11)$$

The other coefficient is

$$a_\alpha = \langle \, \delta v_\alpha \, \rangle \, . \quad (3.12)$$

It is known as the coefficient of **dynamic friction**.

A Brownian particle moving with velocity $\mathbf{v}$ through the gas experiences a drag opposing the motion (Fig. 1.4).

In order to find the mean values appearing in the Fokker-Planck equation, we have to make clear the **physical and mathematical sense** of expressions (3.11) and (3.12).

> The mean values of velocity changes are in fact **statistically averaged** and determined by the forces acting between a test particle and scatterers (**field** particles or waves).

For test particles interacting with the **thermal electrons and ions** in a plasma, such calculations give us the **Landau integral**.

Thus one did not anticipate any major problems in rewriting the Landau integral in the Fokker-Planck form.

The kinetic equation found in this way will allow us to study the Coulomb interaction of accelerated particle beams with astrophysical plasma.

Collisional friction slows down the particles of the beam and moves them toward the zero velocity in the velocity space (Fig. 3.2).

Diffusion **disperses** the distribution of beam particles in the velocity space.

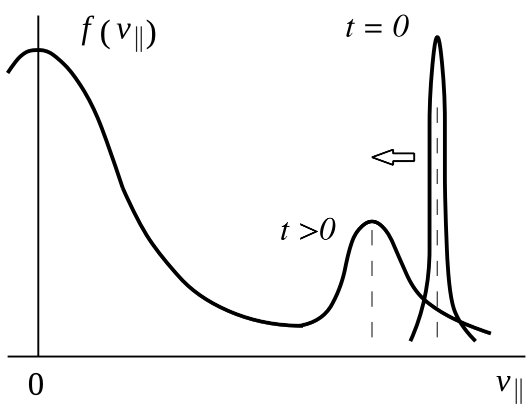


Figure 3.2: A beam of fast particles in plasma. We illustrate only the effects of Coulomb collisions.

During the motion of a beam of fast particles in a plasma a **reverse** current of thermal electrons is generated, which tends to compensate the electric current of fast particles – the **direct** current.

> The electric field driving the reverse current makes a great impact on the particle beam kinetics.

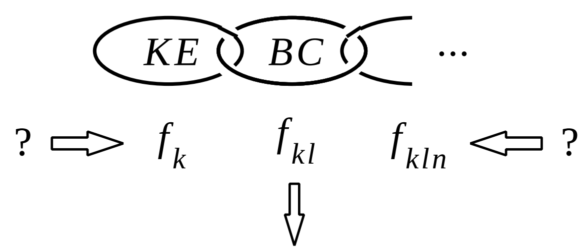
That is why, in order to solve the problem of accelerated particle propagation in, for example, the solar atmosphere, we inevitably have to apply a **combined approach**.

This takes into account both the electric field influence on the accelerated particles (as in the Vlasov equation) and their scattering from the thermal particles of a plasma.

3.2 Correlations and Debye-Hückel shielding

We are going to understand the most fundamental property of the binary **correlation function**.

With this aim, we shall solve the **second** equation in the chain illustrated by Fig.



Here BC is the Equation (2.45) for the correlation function f_{kl}.

To determine and to solve this equation we have to know **two functions**:

the distribution function f_k from the **first** link in the chain and

the triple correlation function f_{kln} from the **third** link.

3.2.1 The Maxwellian distribution function

Let us consider the **stationary** ($\partial/\partial t = 0$) solution to the equations for correlation functions, assuming the interaction parameter ζ_{i} to be small and using the **successive approximations** in the following form.

First, we set

$$f_{kl} = 0$$

in the kinetic equation.

Second, we assume that the triple correlation function

$$f_{kln} = 0$$

in Equation (2.45) for the correlation function f_{kl} etc.

The plasma is supposed to be **stationary, uniform** and in the **thermodynamic equilibrium** state, i.e. the velocity distribution is assumed to be a Maxwellian function

$$f_k\,(X) = f_k\,(v^2) = c_k\,\exp\left(-\frac{m_k\,v^2}{2k_{\mathrm{B}}T_k}\right). \qquad (3.13)$$

The constant c_k is determined by the normalizing condition and equals

$$c_k = n_k\left(\frac{m_k}{2\pi\,k_{\mathrm{B}}T_k}\right)^{3/2}.$$

It is obvious that the Maxwellian function satisfies the kinetic equation under assumptions made above **if the averaged force is equal to zero**:

$$F_{k,\alpha}(X,t) = F_{k,\alpha}(X) = 0\,. \tag{3.14}$$

Since we shall need the same assumption in the next Section, we shall justify it there.

3.2.2 The averaged force and electric neutrality

Let us substitute the Maxwellian function in the kinetic equation, neglecting all the interactions except the Coulomb ones.

We obtain the following expression for the averaged force:

$$F_{k,\alpha}\left(X_1\right) = \sum_l \int\limits_{X_2} F_{kl,\alpha}\left(X_1, X_2\right)\, f_l\left(X_2\right) dX_2 =$$

since plasma is **uniform**, f_l does not depend of $\mathbf{r}_2$

$$\begin{aligned} &= \sum_l \int\limits_{\mathbf{r}_2} F_{kl,\alpha}\left(\mathbf{r}_1, \mathbf{r}_2\right) d^3\mathbf{r}_2 \cdot \int\limits_{\mathbf{v}_2} f_l\left(\mathbf{v}_2\right) d^3\mathbf{v}_2 = \\ &= -\int\limits_{\mathbf{r}_2} \sum_l \frac{\partial}{\partial r_{1,\alpha}} \left(\frac{e_k\, e_l}{|\,\mathbf{r}_1 - \mathbf{r}_2\,|}\right) d^3\mathbf{r}_2 \cdot n_l = \\ &= -\int\limits_{\mathbf{r}_2} \frac{\partial}{\partial r_{1,\alpha}} \left(\frac{e_k}{|\,\mathbf{r}_1 - \mathbf{r}_2\,|}\right) d^3\mathbf{r}_2 \cdot \sum_l n_l\, e_l\,. \end{aligned} \tag{3.15}$$

Therefore

$$F_{k,\alpha} = 0\,, \tag{3.16}$$

if the plasma is assumed to be **electrically neutral**:

$$\boxed{\sum_l n_l\, e_l = 0\,.} \tag{3.17}$$

Balanced charges of ions and electrons determine the name **plasma** according Langmuir (1928).

So the averaged (statistically smoothed) force (2.32) is equal to zero in the electrically neutral plasma but is **not** equal to zero in a system of charged particles of the same charge sign: positive or negative, it does not matter.

Such a system tends to **expand**.

There is **no** neutrality in **gravitational** systems like **stellar clusters**.

The large-scale gravitational field makes an **overall thermodynamic** equilibrium impossible.

Moreover, on the contrary to plasma, they tend to contract and **collapse**.

3.2.3 Pair correlations and the Debye-Hückel radius

As a first approximation, on putting the triple correlation function

$$f_{kln} = 0\,,$$

we obtain from Equation (2.45), in view of condition (3.16), the following equation for the binary **correlation function** f_{kl}:

$$v_{1,\alpha}\,\frac{\partial f_{kl}}{\partial\, r_{1,\alpha}} + v_{2,\alpha}\,\frac{\partial f_{kl}}{\partial\, r_{2,\alpha}} =$$

$$= -\sum_n \int\limits_{X_3} \left\{ \frac{1}{m_k}\, F_{kn,\alpha}\,(X_1, X_3)\; f_{nl}\,(X_3, X_2)\; \frac{\partial f_k}{\partial\, v_{1,\alpha}} + \right.$$

$$\left. + \frac{1}{m_l}\, F_{ln,\alpha}\,(X_2, X_3)\; f_{nk}\,(X_3, X_1)\; \frac{\partial f_l}{\partial\, v_{2,\alpha}} \right\} dX_3\,. \quad (3.18)$$

Let us consider the particles of two kinds: electrons and ions, assuming the ions to be **motionless** and **homogeneously** distributed.

Then the ions do **not** take part in any kinetic processes.

Hence

$$\hat{\varphi}_{\,\mathrm{i}} \equiv 0$$

for ions; and the correlation functions associated with $\hat{\varphi}_{\,\mathrm{i}}$ equal zero too:

$$f_{\mathrm{ii}} = 0\,, \qquad f_{\mathrm{ei}} = 0 \qquad \text{etc.} \qquad (3.19)$$

Among the pair correlation functions, **only one** has a non-zero magnitude

$$f_{\mathrm{ee}}\left(X_1, X_2\right) = f\left(X_1, X_2\right). \tag{3.20}$$

Taking into account (3.19), (3.20), and (3.13), rewrite Equation (3.18) as follows

$$\begin{aligned} &\mathbf{v}_1 \frac{\partial f}{\partial \mathbf{r}_1} + \mathbf{v}_2 \frac{\partial f}{\partial \mathbf{r}_2} = \\ &= \frac{1}{k_{\mathrm{B}} T} \int\limits_{X_3} \left[\mathbf{v}_1 \cdot \mathbf{F}\left(X_1, X_3\right) f\left(X_3, X_2\right) f_{\mathrm{e}}\left(\mathbf{v}_1\right) + \right. \\ &\left. + \mathbf{v}_2 \cdot \mathbf{F}\left(X_2, X_3\right) f\left(X_1, X_3\right) f_{\mathrm{e}}\left(\mathbf{v}_2\right) \right] dX_3 . \end{aligned} \tag{3.21}$$

Since $\mathbf{v}_1$ and $\mathbf{v}_2$ are arbitrary and refer to the same kind of particles (electrons), (3.21) takes the form

$$\frac{\partial f}{\partial \mathbf{r}_1} = \frac{1}{k_{\mathrm{B}} T} \int\limits_{X_3} \mathbf{F}\left(X_1, X_3\right) f\left(X_3, X_2\right) f_{\mathrm{e}}\left(\mathbf{v}_1\right) dX_3 . \tag{3.22}$$

Taking into account the **Coulomb force** in the same approximation as (3.16) and assuming the correlation to exist only **between the positions** of the particles in space (rather than between velocities), we integrate both sides of (3.22) over $d^3\mathbf{v}_1\, d^3\mathbf{v}_2$.

The result is

$$\frac{\partial\, g\,(\mathbf{r}_1, \mathbf{r}_2)}{\partial\, \mathbf{r}_1} = -\frac{ne^2}{k_{\mathrm{B}} T} \int\limits_{\mathbf{r}_3} \nabla_{\mathbf{r}_1} \frac{1}{|\, \mathbf{r}_1 - \mathbf{r}_3\, |}\; g\,(\mathbf{r}_2, \mathbf{r}_3)\; d^3\mathbf{r}_3\,. \tag{3.23}$$

Here the function

$$g\,(\mathbf{r}_1, \mathbf{r}_2) = \int\limits_{\mathbf{v}_1} \int\limits_{\mathbf{v}_2} f\,(X_1, X_2)\; d^3\mathbf{v}_1\, d^3\mathbf{v}_2\,. \tag{3.24}$$

We integrate Equation (3.23) over $\mathbf{r}_1$ and designate the function

$$g\,(\mathbf{r}_1, \mathbf{r}_2) = g\,(r_{12}^2)\,,$$

where

$$r_{12} = |\, \mathbf{r}_1 - \mathbf{r}_2\, |\,.$$

So we obtain the equation

$$g\,(r_{12}^2) = -\frac{ne^2}{k_{\mathrm{B}} T} \int\limits_{\mathbf{r}_3} \frac{g\,(r_{23}^2)}{r_{13}}\; d^3\mathbf{r}_3\,.$$

Its **solution** is

$$g\,(r) = \frac{c_0}{r}\, \exp\left(-\frac{r}{r_{\mathrm{DH}}}\right), \tag{3.25}$$

where

$$\boxed{r_{\mathrm{DH}} = \left(\frac{k_{\mathrm{B}} T}{4\pi\, ne^2}\right)^{1/2}} \tag{3.26}$$

is the **Debye-Hückel radius** or, more exactly, the **electron** Debye-Hückel radius.

The constant of integration

$$c_0 = -\frac{1}{4\pi\, r_{\rm DH}^2\, n} \tag{3.27}$$

(Exercise 3.8).

Substituting (3.27) in solution (3.25) gives the sought-after pair correlation function

$$g\,(r) = -\frac{1}{k_{\rm B}T}\,\frac{e^2}{r}\,\exp\left(-\frac{r}{r_{\rm DH}}\right). \tag{3.28}$$

This formula shows that

> the Debye-Hückel radius is a characteristic length of the pair correlations in a fully-ionized equilibrium plasma.

As one might have anticipated,

> the binary correlation function reproduces the shape of the **actual** potential of interaction, i.e. the **shielded** Coulomb potential:

$$\boxed{g\,(r) \sim \varphi\,(r) \sim \frac{1}{r}\,\exp\left(-\frac{r}{r_{\rm DH}}\right)\,.} \tag{3.29}$$

Astrophysical plasmas exhibit **collective phenomena** arising out of mutual interactions of many particles.

Since the radius $r_{\rm DH}$ is a characteristic length of pair correlations, the number $n\, r_{\rm DH}^3$ gives us a measure of the number of particles which can **interact simultaneously**.

The inverse of this number is the so-called **plasma parameter**

$$\zeta_{\rm p} = \left(n\, r_{\rm DH}^{3}\right)^{-1} . \tag{3.30}$$

This is a small quantity as well as it can be expressed in terms of the interaction parameter $\zeta_{\rm i}$ (Exercise 3.1).

In many astrophysical applications, the plasma parameter is really small.

Thus, the number of particles inside the Debye-Hückel sphere is very large (Exercise 3.2).

So

> the collective phenomena can be really important in astrophysical plasma in many places where it is weakly coupled.

3.3 Gravitational systems

A **fundamental difference** between the astrophysical plasmas and the gravitational systems lies in the nature of the gravitational force:

there is **no shielding** to vitiate this long-range $1/r^2$ force.

The collisional integral formally equals infinity.

The **conventional wisdom** of such systems asserts that they can be described by the **collisionless** kinetic equation, the **gravitational analog** of the Vlasov equation.

Comment:

$$\infty \quad ==> \quad 0 \quad !!!$$

On the basis of what we have seen above,

> the collisionless approach in gravitational systems, i.e. the entire neglect of particle pair correlations, constitutes an **uncontrolled approximation**.

Unlike the plasma, we cannot derive the next order correction to the collisionless equation in the perturbation expansion.

We may hope to circumvent this difficulty by **first** identifying the **mean field** force $\langle \mathbf{F} \rangle$, acting at any given point in space and **then** treating **fluctuations** $\mathbf{F}'$ away from the mean field force.

However this is difficult to implement concretely because of the apparent **absence of a clean separation** of scales.

3.4 Comments on numerical simulations

The astrophysical plasma processes are typically investigated in **well developed** and distinct **approaches**.

One approach, described by the Vlasov equation, is the **collisionless** limit used when collective effects dominate.

In cases where the plasma dynamics is determined by **collisional** processes and where the self-consistent fields can be neglected, the Fokker-Planck approach is used.

At the same time, it is known that

> both collective effects and Coulomb collisions can play an essential role in a great variety of astrophysical phenomena.

Besides, **collisions play the principal role in the physics of collisionless plasma.**

Taking collisions into account may lead not only to quantitative but also qualitative changes in the plasma behavior.

Even in the **collisionless** limit, the kinetic equation is difficult for numerical simulations, and the '**macroparticle**' method is widely used algorithms.

Instead of direct numerical solution of the kinetic equation, a set of **ordinary** differential equations for every macroparticle is solved.

These equations are the characteristics of the Vlasov equation.

In the case of a **collisional** plasma, the position of a macroparticle satisfies the usual equation of the collisionless case

$$\dot{\mathbf{r}} \equiv \frac{d\,\mathbf{r}}{dt} = \mathbf{v}(t)\,. \tag{3.31}$$

However the momentum equation is modified owing to the Coulomb collisions.

They are described by the Fokker-Planck operator (3.10) which introduces a **friction** and **diffusion** in velocity space.

Thus it is necessary to find the **effective collisional force** $\mathbf{F}_{\mathrm{c}}$ which acts on the macroparticles:

$$\dot{\mathbf{v}} \equiv \frac{d\,\mathbf{v}}{dt} = \frac{1}{m}\left(\mathbf{F}_{\mathrm{L}} + \mathbf{F}_{\mathrm{c}}\right)\,. \tag{3.32}$$

The collisional force can be introduced phenomenologically but a more mathematically correct approach can be constructed using the stochastic equivalence of the Fokker-Planck and **Langevin equations**.

Stochastic differential equations are regarded as an alternative to the description of astrophysical plasma in terms of distribution function.

> The Langevin approach allows one to overcome difficulties related to the Fokker-Planck equation and to simulate actual plasma processes, taking account of both collective effects and Coulomb collisions.

Generally, if we construct a method for the simulation of complex processes in astrophysical plasma, we have to satisfy the following obvious but **conflicting conditions**.

First, the method should be adequate for the task in hand.

For a number of problems the **simplified models** of the collisional integral can provide a correct description and ensure a desired accuracy.

Second, the method should be computationally efficient.

The algorithm should **not** be extremely time-consuming.

In practice, some **compromise** between **accuracy and complexity** of the method should be achieved.

A '**recipe**':

> the choice of a particular model of the collisional integral is determined by the importance and particular features of the collisional processes in a given astrophysical problem.

3.5 Practice: Exercises and Answers

Exercise 3.1. Show that the interaction parameter $\zeta_{\rm i}$ is related to the plasma parameter $\zeta_{\rm p}$ as follows:

$$\zeta_{\rm i} = \frac{1}{4\pi}\,\zeta_{\rm p}^{\,2/3}\,. \tag{3.33}$$

Exercise 3.2. How many particles are inside the Debye-Hückel sphere in plasma of the solar corona?

Answer.

For an electron-proton plasma with $T \approx 2 \times 10^6\,\mathrm{K}$ and $n \approx 2 \times 10^8\,\mathrm{cm}^{-3}$, the Debye-Hückel radius

$$r_{_{\rm DH}} = \left(\frac{kT}{8\pi\, e^2\, n} \right)^{1/2} \approx 4.9 \left(\frac{T}{n} \right)^{1/2} \approx 0.5 \text{ cm}\,. \qquad (3.34)$$

The number of particles inside the Debye-Hückel sphere

$$N_{_{\rm DH}} = n\, \frac{4}{3}\, \pi r_{_{\rm DH}}^3 \sim 10^8. \qquad (3.35)$$

Hence the typical value of plasma parameter in the corona is really small:

$$\zeta_{\rm p} \sim 10^{-8}\,.$$

The interaction parameter is also small:

$$\zeta_{\rm i} \sim 10^{-6}\,.$$

Exercise 3.3. Estimate the interaction parameter (3.2) in the interior of **white dwarf** stars (see also Exercise 1.3).

Exercise 3.4. Let $w = w\,(\mathbf{v},\, \delta\mathbf{v})$ be the probability that a test particle changes its velocity $\mathbf{v}$ to $\mathbf{v} + \delta\mathbf{v}$ in the time interval δt.

The velocity distribution at the time t can be written as

$$f(\mathbf{v}, t) = \int f(\mathbf{v} - \delta\mathbf{v},\, t - \delta t)\, w\,(\mathbf{v} - \delta\mathbf{v},\, \delta\mathbf{v})\, d^3\delta\mathbf{v}\,. \quad (3.36)$$

Show that

> the Fokker-Planck equation follows from the Taylor series expansion of the function $f(\mathbf{v}, t)$ given by formula (3.36).

Exercise 3.5. Express the collisional integral in terms of the differential cross-sections of interaction between particles.

Exercise 3.6. Show that

> the Fokker-Planck collisional model can be derived from the Boltzmann collisional integral

under the assumption that the change in the velocity of a particle due to a collision is rather small.

Exercise 3.7. The Landau integral is generally thought to approximate the Boltzmann integral for the 1/r potential to a '**dominant order**', i.e. to within terms of order $1/\ln\Lambda$, where $\ln\Lambda$ is the Coulomb logarithm.

However this is not the whole truth.

Show that

> the Landau integral approximates the Boltzmann integral to the dominant order only in parts of the velocity space.

Exercise 3.8. Find the constant of integration c_0 in formula (3.25).

Exercise 3.9. Write and discuss the gravitational analog of the Vlasov equation.

Answer.

The basic assumption is that the gravitational N-body system can be described in terms of a statistically smooth distribution function $f(X, t)$.

The Vlasov equation manifests that this function will stream freely in the self-consistent gravitational potential $\phi(\mathbf{r}, t)$ associated with $f(X, t)$, so that

$$\frac{\partial f(X,t)}{\partial t} + v_\alpha \frac{\partial f(X,t)}{\partial r_\alpha} - \frac{\partial \phi}{\partial r_\alpha} \frac{\partial f(X,t)}{\partial v_\alpha} = 0\,. \tag{3.37}$$

Here

$$\Delta \phi = -\,4\pi\, G \rho(\mathbf{r}, t) \tag{3.38}$$

and

$$\rho(\mathbf{r}, t) = \int f(\mathbf{r}, \mathbf{v}, t)\, d^3\mathbf{v}\,. \tag{3.39}$$

Note that, in the context of the mean field theory, a distribution of particles over their masses has no effect.

Applying for example to the system of stars in a galaxy, Equation (3.37) implies that

> the net gravitational force acting on a star is determined by the large-scale structure of the galaxy rather than by whether the star happens to lie close to some other star.

The force acting on any star does not vary rapidly, and each star is supposed to accelerate smoothly through the force field generated by the galaxy as a whole.

In fact, **gravitational encounters are not screened**, they can be thought of as leading to an additional collisional term on the right side of the equation – a **collisional** integral.

However very little is known mathematically about such possibility.

Exercise 3.10. Discuss a gravitational analog of the Landau integral in the following form

$$\left(\frac{\partial \hat{f}}{\partial t}\right)_{\mathrm{c}} = \sigma \, \frac{\partial}{\partial \mathbf{v}} \int\limits_{\mathbf{v}'} \frac{\partial^2 \,|\, \mathbf{v} - \mathbf{v}' \,|}{\partial \mathbf{v} \, \partial \mathbf{v}'} \cdot \left(\frac{\partial}{\partial \mathbf{v}} - \frac{\partial}{\partial \mathbf{v}'}\right) \times$$
$$\times \left[\, f(\mathbf{r},\, \mathbf{v},\, t)\, f(\mathbf{r},\, \mathbf{v}',\, t) \,\right] d^3\mathbf{v}' \, . \qquad (3.40)$$

Here σ is a constant determined by the effective collision rate.

Chapter 4

Macroscopic Description of Astrophysical Plasma

> In this Chapter we treat individual kinds of particles as **continuous media**, mutually penetrating charged gases which interact between themselves and with an electromagnetic field.
>
> This approach gives us the **multi-fluid model** which is useful to consider many properties of astrophysical plasmas, e.g., the solar wind.

4.1 Summary of microscopic description

The kinetic equation gives us a **microscopic** (though averaged in a statistical sense) description of plasma.

Let us consider the transition to a less comprehensive **macroscopic** description.

We start from the kinetic equation for particles of kind k

$$\frac{\partial f_k(X,t)}{\partial t} + v_\alpha \frac{\partial f_k(X,t)}{\partial r_\alpha} +$$

$$+ \frac{F_{k,\alpha}(X,t)}{m_k} \frac{\partial f_k(X,t)}{\partial v_\alpha} = \left(\frac{\partial \hat{f}_k}{\partial t}\right)_{\mathrm{c}} . \tag{4.1}$$

Here the **statistically averaged** force is

$$F_{k,\alpha}(X,t) = \sum_l \int_{X_1} F_{kl,\alpha}(X,X_1)\ f_l(X_1,t)\ dX_1 \tag{4.2}$$

and the **collisional integral**

$$\left(\frac{\partial \hat{f}_k}{\partial t}\right)_{\mathrm{c}} = - \frac{\partial}{\partial v_\alpha} J_{k,\alpha}(X,t) , \tag{4.3}$$

where the **flux of particles** of kind k

$$J_{k,\alpha}(X,t) = \sum_l \int_{X_1} \frac{1}{m_k} F_{kl,\alpha}(X,X_1)\ f_{kl}(X,X_1,t)\ dX_1 \tag{4.4}$$

in the 6D phase space $X = \{\mathbf{r},\mathbf{v}\}$.

4.2 Definition of macroscopic quantities

Before the deduction of equations for the macroscopic quantities or macroscopic **transfer** equations, let us define the following **moments** of the distribution function.

(a) The zeroth moment (without multiplying the distribution function f_k by the velocity $\mathbf{v}$)

$$\int_{\mathbf{v}} f_k\,(\mathbf{r},\mathbf{v},t)\,d^3\mathbf{v} = n_k(\mathbf{r},t) \tag{4.5}$$

is the number of particles of kind k in a unit volume.

It is related to the **mass density** of particles of kind k

$$\rho_k(\mathbf{r},t) = m_k\, n_k(\mathbf{r},t)\,.$$

The plasma mass density is accordingly

$$\rho\,(\mathbf{r},t) = \sum_k m_k\, n_k(\mathbf{r},t)\,. \tag{4.6}$$

(b) The first moment of the distribution function, i.e. the integral of the product of the velocity $\mathbf{v}$ to the first power and the distribution function f_k,

$$\int_{\mathbf{v}} v_\alpha\, f_k(\mathbf{r},\mathbf{v},t)\,d^3\mathbf{v} = n_k\, u_{k,\alpha} \tag{4.7}$$

is the particle flux, i.e. product of the number density by their **mean velocity**

$$u_{k,\alpha}(\mathbf{r},t) = \frac{1}{n_k}\int_{\mathbf{v}} v_\alpha\, f_k(\mathbf{r},\mathbf{v},t)\,d^3\mathbf{v}\,. \tag{4.8}$$

Consequently, the **mean momentum** of particles of kind k in a unit volume is expressed as follows

$$m_k\, n_k\, u_{k,\alpha} = m_k \int_{\mathbf{v}} v_\alpha\, f_k(\mathbf{r},\mathbf{v},t)\,d^3\mathbf{v}\,. \tag{4.9}$$

(c) The second moment of the distribution function is defined to be

$$\Pi_{\alpha\beta}^{(k)}(\mathbf{r},t) = m_k \int_{\mathbf{v}} v_\alpha v_\beta \; f_k(\mathbf{r},\mathbf{v},t)\; d^3\mathbf{v} =$$

$$= m_k n_k \, u_{k,\alpha} u_{k,\beta} + p_{\alpha\beta}^{(k)} \,. \qquad (4.10)$$

Here we have introduced

$$v'_\alpha = v_\alpha - u_{k,\alpha}$$

which is the deviation of the particle velocity from its mean value (4.8), so that

$$\langle\, v'_\alpha \,\rangle_v = 0\,;$$

and

$$p_{\alpha\beta}^{(k)} = m_k \int_{\mathbf{v}} v'_\alpha v'_\beta \; f_k(\mathbf{r},\mathbf{v},t)\; d^3\mathbf{v} \;, \qquad (4.11)$$

is the **pressure tensor**.

$\Pi_{\alpha\beta}^{(k)}$ is the **tensor of momentum flux** for particles of kind k.

Its component $\Pi_{\alpha\beta}^{(k)}$ is the αth component of the momentum transported by the particles of kind k, in a unit time, across the unit area perpendicular to the axis r_β.

Once we know the distribution function $f_k(\mathbf{r},\mathbf{v},t)$, we can derive all **macroscopic** quantities related to these particles.

So, higher moments of the distribution function will be introduced as needed.

4.3 Macroscopic transfer equations

Note that the deduction of macroscopic equations is just derivation of the equations for the distribution function moments.

4.3.1 Equation for the zeroth moment

Let us calculate the **zeroth** moment of the kinetic equation:

$$\int\limits_{\mathbf{v}} \frac{\partial f_k}{\partial t}\, d^3\mathbf{v} + \int\limits_{\mathbf{v}} v_\alpha \frac{\partial f_k}{\partial r_\alpha}\, d^3\mathbf{v} +$$

$$+ \int\limits_{\mathbf{v}} \frac{F_{k,\alpha}}{m_k} \frac{\partial f_k}{\partial v_\alpha}\, d^3\mathbf{v} = \int\limits_{\mathbf{v}} \left(\frac{\partial \hat{f}_k}{\partial t} \right)_{\mathrm{c}} d^3\mathbf{v}\,. \qquad (4.12)$$

We interchange the order of integration over velocities and the differentiation with respect to time t in the first term and with respect to coordinates r_α in the second one.

Under the second integral

$$v_\alpha \frac{\partial f_k}{\partial r_\alpha} = \frac{\partial}{\partial r_\alpha}(v_\alpha f_k) - f_k \frac{\partial v_\alpha}{\partial r_\alpha} = \frac{\partial}{\partial r_\alpha}(v_\alpha f_k) - 0\,,$$

since **r** and **v** are **independent** variables in the phase space X.

Taking into account that the distribution function **quickly approaches zero as** $v \to \infty$, the integral of the third term is taken by parts and equals zero (Exercise 4.1).

The integral of the right-hand side of (4.12) describes the **change in the number of particles** of kind k as a result of collisions with particles of other kinds.

If the processes of transformation, during which the particle kind can be changed (such as ionization, recombination, charge exchange etc., see Exercise 4.2), are not allowed for, then the last integral is zero as well:

$$\int_{\mathbf{v}} \left(\frac{\partial \hat{f}_k}{\partial t} \right)_{\mathrm{c}} d^3\mathbf{v} = 0\,. \tag{4.13}$$

Thus, by integration of (4.12), the following equation is found

$$\boxed{\frac{\partial n_k}{\partial t} + \frac{\partial}{\partial r_\alpha}\, n_k\, u_{k,\alpha} = 0\,.} \tag{4.14}$$

This is the **continuity equation** expressing the conservation of particles of kind k or (i.e. the same, of course) conservation of their mass:

$$\frac{\partial \rho_k}{\partial t} + \frac{\partial}{\partial r_\alpha}\, \rho_k\, u_{k,\alpha} = 0\,. \tag{4.15}$$

> Equation (4.14) for the zeroth moment n_k depends on the **unknown first moment** $u_{k,\alpha}$.

This is illustrated by Fig. 4.1.

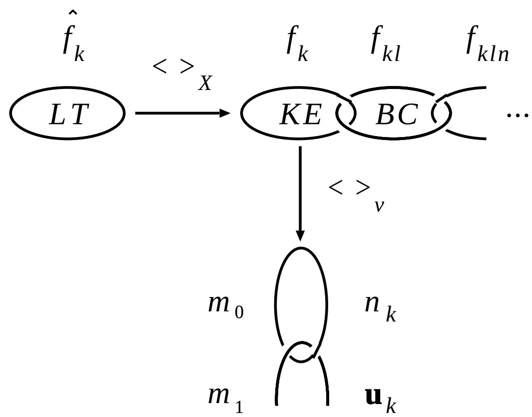


Figure 4.1: KE is the kinetic equation, m_0 is the equation for the zeroth moment of the distribution function f_k.

4.3.2 The momentum conservation law

Now let us calculate the **first** moment of the kinetic equation multiplied by the mass m_k:

$$m_k \int\limits_{\mathbf{v}} \frac{\partial f_k}{\partial t} v_\alpha \; d^3\mathbf{v} +$$

$$+ m_k \int\limits_{\mathbf{v}} v_\alpha v_\beta \frac{\partial f_k}{\partial r_\beta} \; d^3\mathbf{v} + \int\limits_{\mathbf{v}} v_\alpha \, F_{k,\beta} \frac{\partial f_k}{\partial v_\beta} \; d^3\mathbf{v} =$$

$$= m_k \int\limits_{\mathbf{v}} v_\alpha \left(\frac{\partial \hat{f}_k}{\partial t} \right)_{\mathrm{c}} d^3\mathbf{v} \, . \tag{4.16}$$

With allowance made for the definitions (4.7) and (4.10), we obtain the **momentum conservation law**

$$\frac{\partial}{\partial t} (m_k n_k \, u_{k,\alpha}) + \frac{\partial}{\partial r_\beta} \left(m_k n_k \, u_{k,\alpha} u_{k,\beta} + p^{(k)}_{\alpha\beta} \right) -$$

$$- \langle\, F_{k,\alpha}(\mathbf{r},t)\,\rangle_v = \langle\, F^{(\mathrm{c})}_{k,\alpha}\,(\mathbf{r},t)\,\rangle_v\,. \tag{4.17}$$

Here $p^{(k)}_{\alpha\beta}$ is the pressure tensor (4.11), i.e. a part of the unknown second moment (4.10).

The **mean force** acting on the particles of kind k in a unit volume is

$$\langle\, F_{k,\alpha}\,(\mathbf{r},t)\,\rangle_v = \int\limits_{\mathbf{v}} F_{k,\alpha}\,(\mathbf{r},\mathbf{v},t)\; f_k\,(\mathbf{r},\mathbf{v},t)\; d^3\mathbf{v}\,. \tag{4.18}$$

In the case of the Lorentz force, the mean force

$$\langle\, F_{k,\alpha}\,(\mathbf{r},t)\,\rangle_v = n_k e_k \left[\, E_\alpha + \frac{1}{c}\,(\,\mathbf{u}_k \times \mathbf{B}\,)_\alpha \right]$$

or

$$\boxed{\langle\, F_{k,\alpha}\,(\mathbf{r},t)\,\rangle_v = \rho^{\mathrm{q}}_k\, E_\alpha + \frac{1}{c}\,(\,\mathbf{j}^{\,\mathrm{q}}_k \times \mathbf{B}\,)_\alpha\,.} \tag{4.19}$$

Here ρ^{q}_k and $\mathbf{j}^{\,\mathrm{q}}_k$ are the mean densities of electric charge and current, produced by the particles of kind k.

Note that

> the mean electromagnetic force couples all the charged components of astrophysical plasma together

because the electric and magnetic fields, $\mathbf{E}$ and $\mathbf{B}$, act on **all charged components** and, at the same time, all charged components contribute to the electric and magnetic fields according to Maxwell's equations.

The right-hand side of Equation (4.17) contains the mean force resulting from collisions, the **mean collisional force**

$$\langle F_{k,\alpha}^{(c)}(\mathbf{r},t)\rangle_v = m_k \int\limits_{\mathbf{v}} v_\alpha \left(\frac{\partial \hat{f}_k}{\partial t}\right)_c d^3\mathbf{v}\,. \qquad (4.20)$$

Substituting (4.3) in definition (4.20) and integrating gives us the most general formula for the mean collisional force

$$\langle F_{k,\alpha}^{(c)}(\mathbf{r},t)\rangle_v = m_k \int\limits_{\mathbf{v}} J_{k,\alpha}(\mathbf{r},\mathbf{v},t)\, d^3\mathbf{v} = \qquad (4.21)$$

$$= \sum_{l\neq k} \int\limits_{\mathbf{v}} \int\limits_{\mathbf{v_1}} \int\limits_{\mathbf{r_1}} F_{kl,\alpha}(\mathbf{r},\mathbf{v},\mathbf{r}_1,\mathbf{v}_1)\; f_{kl}(\mathbf{r},\mathbf{v},\mathbf{r}_1,\mathbf{v}_1,t)\; d^3\mathbf{r}_1\, d^3\mathbf{v}_1\, d^3\mathbf{v}\,.$$

Note that

> for the particles of the same kind, the **elastic** collisions cannot change the total particle momentum per unit volume.

That is why $l \neq k$ in the sum (4.21).

Formula (4.21) contains the **unknown** binary correlation function f_{kl}.

The last should be found from the correlation function Equation (2.45) indicated as the second link BC in Fig. 4.2.

Thus

the equation for the first moment of the distribution function is as much **unclosed** as the initial kinetic equation.

Therefore the equation for the first moment is unclosed in **two directions**.

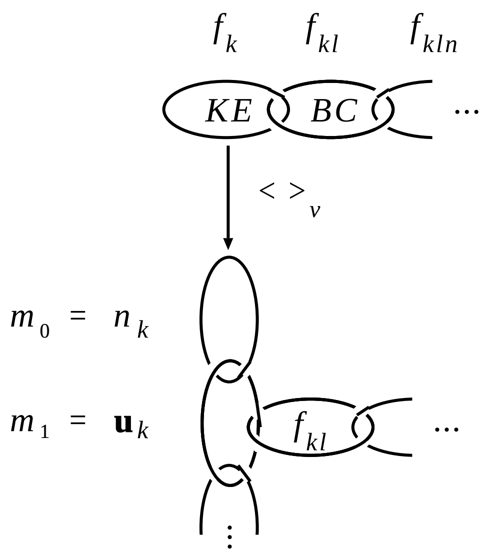


Figure 4.2: m_0, m_1 are the equations for the first two moments. The link m_1 is unclosed in two directions.

If each of kinds of particles is in thermodynamic equilibrium, then the mean collisional force can be expressed in terms of the **mean momentum loss** during the collisions of a particle of kind k with the particles of other kinds:

$$\boxed{\langle\, F^{(\mathrm{c})}_{k,\alpha}\,(\mathbf{r}, t)\,\rangle_{\,v} \;=\; -\sum_{l\neq k} \frac{m_k n_k\,(u_{k,\alpha} - u_{l,\alpha})}{\tau_{kl}}\;.} \tag{4.22}$$

Here $\tau_{kl}^{-1} = \nu_{kl}$ is the mean frequency of collisions between the particles of kinds k and l.

If $u_{k,\alpha} > u_{l,\alpha}$ then the mean collisional force is negative:

> the fast particles of kind k slow down by collisions with the slowly moving particles of other kinds.

The force is zero, once the particles of all kinds have identical mean velocities.

Therefore

> the mean collisional force, as well as the mean electromagnetic force, tends to make astrophysical plasma be a **single** hydrodynamic medium.

4.4 The energy conservation law

4.4.1 The second moment equation

The second moment of a distribution function f_k is the tensor of **momentum flux** density

$$\Pi_{\alpha\beta}^{(k)}(\mathbf{r}, t) = m_k \int\limits_{\mathbf{v}} v_\alpha v_\beta \; f_k(\mathbf{r}, \mathbf{v}, t) \; d^3\mathbf{v} =$$

$$= m_k n_k \, u_{k,\alpha} u_{k,\beta} + p_{\alpha\beta}^{(k)} \, .$$

In order to find an equation for this tensor, we should multiply the kinetic equation

$$\frac{\partial f_k}{\partial t} + v_\alpha \frac{\partial f_k}{\partial r_\alpha} + \frac{F_{k,\alpha}}{m_k} \frac{\partial f_k}{\partial v_\alpha} = \left(\frac{\partial \hat{f}_k}{\partial t} \right)_{\mathrm{c}}$$

by the factor $m_k \, v_\alpha v_\beta$ and integrate over velocity space $\mathbf{v}$.

In this way, we could arrive to a **matrix equation** in partial derivatives.

If we take the trace of this equation we obtain the partial differential **scalar equation** for energy density of the particles.

This is the correct self-consistent way which is the basis of the **moment method**.

For our aims, a **simpler direct** procedure is sufficient and correct.

In order to derive the **energy conservation law**, we multiply Equation (4.1) by the particle's kinetic energy

$$m_k v_\alpha^2/2$$

and integrate over velocities, taking into account that

$$v_\alpha = u_{k,\alpha} + v_\alpha' \,, \qquad \langle \, v_\alpha' \, \rangle_v = 0 \,,$$

and

$$v_\alpha^2 = u_{k,\alpha}^2 + (v_\alpha')^2 + 2 \, u_{k,\alpha} \, v_\alpha' \,.$$

A straightforward integration yields

$$\frac{\partial}{\partial t} \left(\frac{\rho_k u_k^2}{2} + \rho_k \, \varepsilon_k \right) +$$

$$+ \frac{\partial}{\partial r_\alpha} \left[\rho_k u_{k,\alpha} \left(\frac{u_k^2}{2} + \varepsilon_k \right) + p_{\alpha\beta}^{(k)} \, u_{k,\beta} + q_{\,k,\alpha} \right] =$$

$$= \rho_k^{\mathrm{q}} \left(\mathbf{E} \cdot \mathbf{u}_k \right) + \left(\mathbf{F}_k^{(c)} \cdot \mathbf{u}_k \right) + Q_k^{(c)} \left(\mathbf{r}, t \right) + \mathcal{L}_k^{(r)} \left(\mathbf{r}, t \right) . \quad (4.23)$$

Here

$$m_k \, \varepsilon_k(\mathbf{r}, t) = \frac{1}{n_k} \int\limits_{\mathbf{v}} \frac{m_k \left(v'_\alpha \right)^2}{2} \, f_k \left(\mathbf{r}, \mathbf{v}, t \right) d^3\mathbf{v} =$$

$$= \frac{m_k}{2 n_k} \int\limits_{\mathbf{v}} \left(v'_\alpha \right)^2 \, f_k \left(\mathbf{r}, \mathbf{v}, t \right) d^3\mathbf{v} \quad (4.24)$$

is the **mean kinetic energy** of **chaotic** (non-directed) motion per single particle of kind k.

Thus the first term on the left-hand side of (4.23) represents the time derivative of the energy of the particles of kink k in a unit volume, which is the sum of kinetic energy of a **regular** motion with the mean velocity $\mathbf{u}_k$ and the so-called **internal** energy.

As every tensor, the pressure tensor can be written as

$$p_{\alpha\beta}^{(k)} = p_k \, \delta_{\alpha\beta} + \pi_{\alpha\beta}^{(k)} \, . \quad (4.25)$$

On rearrangement, we obtain the following general equation

$$\frac{\partial}{\partial t} \left(\frac{\rho_k u_k^2}{2} + \rho_k \, \varepsilon_k \right) +$$

$$+ \frac{\partial}{\partial r_\alpha} \left[\rho_k u_{k,\alpha} \left(\frac{u_k^2}{2} + w_k \right) + \, \pi_{\alpha\beta}^{(k)} \, u_{k,\beta} + q_{\,k,\alpha} \right] =$$

$$= \rho_k^{\mathrm{q}} \left(\mathbf{E} \cdot \mathbf{u}_k \right) + \left(\mathbf{F}_k^{(c)} \cdot \mathbf{u}_k \right) + Q_k^{(c)} \left(\mathbf{r}, t \right) + \mathcal{L}_k^{(r)} \left(\mathbf{r}, t \right) . \quad (4.26)$$

Here

$$w_k = \varepsilon_k + \frac{p_k}{\rho_k} \quad (4.27)$$

is the **heat function** per unit mass.

Therefore the second term on the left-hand side contains the energy flux

$$\rho_k u_{k,\alpha} \left(\frac{u_k^2}{2} + w_k \right) ,$$

which can be called the '**advective**' flux of kinetic energy.

Let us mention the known astrophysical application of this term.

The **advective cooling** of ions heated by viscosity might dominate the cooling by the electron-ion collisions, e.g., in a high-temperature plasma flow near a **rotating black hole**.

In an **advection-dominated accretion flow** (ADAF), the heat generated via viscosity is transferred inward the black hole rather than radiated away locally like in a standard accretion disk model.

However, discussing the ADAF as a solution for the important astrophysical problem should be treated with **reasonable cautions**.

Looking at Equations (4.23) for electrons and ions separately, we see that

> too many assumptions have to be made to arrive to the ADAF approximation.

For example, this is not realistic to assume that plasma electrons are heated only due to collisions with ions and, for this reason, the electrons are much cooler than the ions.

The suggestions underlying the ADAF model ignore several effects including **reconnection** and dissipation of magnetic fields (regular and random) in astrophysical plasma.

This makes a physical basis of the ADAF model **uncertain**.

4.4.2 The case of thermodynamic equilibrium

In order to clarify the definitions given above, let us, for a while, come back to the general principles.

If the particles of the kth kind are in the **thermodynamic equilibrium**, then f_k is the Maxwellian function with the **temperature** T_k:

$$f_k^{(0)}(\mathbf{r}, \mathbf{v}) = n_k(\mathbf{r}) \left[\frac{m_k}{2\pi\, k_{\mathrm{B}} T_k(\mathbf{r})} \right]^{3/2} \times$$

$$\times \exp \left\{ - \frac{m_k \,|\, \mathbf{v} - \mathbf{u}_k(\mathbf{r}) \,|^{\,2}}{2\, k_{\mathrm{B}} T_k(\mathbf{r})} \right\} . \qquad (4.28)$$

In this case, according to (4.24), the mean kinetic energy of chaotic motion per single particle of kind k

$$m_k \, \varepsilon_k = \frac{3}{2} \, k_{\rm B} T_k \, . \tag{4.29}$$

The pressure tensor (4.11) is **isotropic**:

$$p_{\alpha\beta}^{(k)} = p_k \, \delta_{\alpha\beta} \, , \tag{4.30}$$

where the scalar

$$p_k = n_k \, k_{\rm B} T_k \tag{4.31}$$

is the **gas pressure** of the particles of kind k.

This is also the equation of state for the **ideal** gas.

Thus we have found that the pressure tensor is **diagonal**.

This implies the absence of **viscosity** for the ideal gas:

$$\pi_{\alpha\beta}^{(k)} = 0 \, . \tag{4.32}$$

The **heat function** per unit mass or, more exactly, the **specific enthalpy** is

$$w_k = \varepsilon_k + \frac{p_k}{\rho_k} = \frac{5}{2} \, \frac{k_{\rm B} T_k}{m_k} \, . \tag{4.33}$$

It was a particular case of the thermodynamic equilibrium.

4.4.3 The general case of anisotropic plasma

In general, we do not expect that the particles of kind k have reached thermodynamic equilibrium.

Nevertheless we often use the mean kinetic energy (4.24) to define the **effective kinetic temperature** T_k according to definition (4.29).

> A **kinetic** temperature is just a measure for the spread of the particle distribution in velocity space.

The kinetic temperatures of different components in astrophysical plasma may differ from each other.

Moreover, in an anisotropic plasma, the kinetic temperatures **parallel** and **perpendicular** to the magnetic field are different.

Without supposing thermodynamic equilibrium, in an **anisotropic** plasma, the part associated with the deviation of the distribution function from the isotropic one is distinguished in the pressure tensor:

$$p_{\alpha\beta}^{(k)} - p_k \, \delta_{\alpha\beta} = \pi_{\alpha\beta}^{(k)} \, . \tag{4.34}$$

Here $\pi_{\alpha\beta}^{(k)}$ is called the **viscous stress tensor**.

Recall that we did not derive an equation for this tensor.

The term $\pi_{\alpha\beta}^{(k)} \, u_{k,\beta}$ in equation (4.23) represents the flux of energy released by the **viscous force** in the particles of kind k.

The last term on the left-hand side of the energy equation, the vector

$$q_{k,\alpha} = \int_{\mathbf{v}} \frac{m_k \, (v')^2}{2} \, v'_\alpha \, f_k \left(\mathbf{r}, \mathbf{v}, t\right) \, d^3\mathbf{v} \tag{4.35}$$

is the **heat flux density** due to the particles of kind k.

Formula (4.35) shows that a **third-order-moment** term appears in the second order moment of the kinetic equation.

The **right-hand side** of the energy conservation law (4.23) contains the following four terms:

(a) The first term

$$\rho_k^{\mathrm{q}} \left(\mathbf{E} \cdot \mathbf{u}_k \right) = n_k e_k \, E_\alpha \, u_{k,\alpha} \tag{4.36}$$

is the work done by the **Lorentz force** (without the magnetic field, of course) in unit time on unit volume.

(b) The second term

$$\left(\mathbf{F}_k^{(\mathrm{c})} \cdot \mathbf{u}_k \right) = u_{k,\alpha} \int_{\mathbf{v}} m_k \, v'_\alpha \left(\frac{\partial \hat{f}_k}{\partial t} \right)_{\mathrm{c}} d^3\mathbf{v} \tag{4.37}$$

is the work done by the collisional **force of friction** of the particles of kind k with all other particles in unit time on unit volume.

> The work of friction force results from the mean momentum change of particles of kind k (*moving with the mean velocity* $\mathbf{u}_k$) owing to collisions with all other particles.

This work equals zero if $\mathbf{u}_k = 0$.

(c) The third term

$$Q_k^{(c)}(\mathbf{r}, t) = \int_{\mathbf{v}} \frac{m_k (v')^2}{2} \left(\frac{\partial \hat{f}_k}{\partial t} \right)_c d^3\mathbf{v} \tag{4.38}$$

is the rate of thermal energy release (heating or cooling) in a gas of the particles of kind k due to collisions with other particles.

Recall that the collisional integral depends on the **binary correlation** function f_{kl}.

(d) The last term $\mathcal{L}_k^{(r)}(\mathbf{r}, t)$ takes into account that a plasma component k can gain energy by absorbing radiations of different kinds and can lose the energy by emitting radiations.

4.5 General properties of transfer equations

4.5.1 Divergent and hydrodynamic forms

Equations (4.14), (4.17), and (4.23) are referred to as the equations of particle, momentum and energy **transfer**.

They are written in the '**divergent**' form.

This essentially states the **conservation laws** and turns out to be convenient in numerical work, to construct the **conservative schemes** for computations.

Sometimes, other forms are more convenient.

For instance, the equation of momentum transfer or simply the equation of motion can be brought into the frequently used form:

$$\rho_k \left(\frac{\partial\, u_{k,\alpha}}{\partial t} + u_{k,\beta} \frac{\partial\, u_{k,\alpha}}{\partial r_\beta} \right) = - \frac{\partial}{\partial r_\beta}\, p^{(k)}_{\alpha\beta} +$$

$$+ \langle\, F_{k,\alpha}\, (\mathbf{r}, t)\, \rangle_{\,v} + \langle\, F^{(\mathrm{c})}_{k,\alpha}\, (\mathbf{r}, t)\, \rangle_{\,v}\, . \qquad (4.39)$$

The so-called **substantial derivative** appears on the left-hand side of this equation:

$$\boxed{\frac{d^{\,(k)}}{dt} = \frac{\partial}{\partial t} + u_{k,\beta}\, \frac{\partial}{\partial r_\beta} = \frac{\partial}{\partial t} + \mathbf{u}_k \cdot \nabla_{\mathbf{r}}\, .} \qquad (4.40)$$

This substantial or **advective** derivative – the total time derivative following a **fluid element** of kind k – is typical of **hydrodynamic-type** equations, to which the equation of motion (4.39) belongs.

> In the frame, in which the fluid element is not moving, the mean velocity $\mathbf{u}_k = 0$ but the time partial derivative $\partial / \partial t$ does not vanish of course.

The total time derivative with respect to the mean velocity $\mathbf{u}_k$ of the particles of kind k is different for each kind k.

In the one-fluid MHD theory, we shall introduce the substantial derivative with respect to the average velocity of the plasma as a whole.

For the case of the **Lorentz force**, the equation of motion of the particles of kind k can be rewritten as follows:

$$\rho_k \, \frac{d^{(k)} \, u_{k,\alpha}}{dt} = -\frac{\partial}{\partial r_\beta} \, p_{\alpha\beta}^{(k)} + \rho_k^{\mathrm{q}} \, E_\alpha + \frac{1}{c} \, (\, \mathbf{j}_k^{\mathrm{q}} \times \mathbf{B} \,)_\alpha + $$

$$+ \, \langle \, F_{k,\alpha}^{(\mathrm{c})} \, (\mathbf{r}, t) \, \rangle_{\,v} \, . \qquad (4.41)$$

Here the right-hand side represents the forces acting on the fluid element of kind k, in particular, the last term is the **mean collisional force**.

The left-hand side of (4.41) is the change of the momentum of this fluid element.

4.5.2 Status of the conservation laws

As we saw above, when we treat a plasma as several **continuous media** (the mutually penetrating charged gases), for each of them,

> the main three average properties (density, velocity, and a quantity like temperature) are governed by the **basic conservation laws** for mass, momentum, and energy in the media.

These conservation equations contain **more unknowns** than the **number of equations**.

The transfer equations for local macroscopic quantities are as much **unclosed** as the initial kinetic equation (see KE in Fig. 4.3).

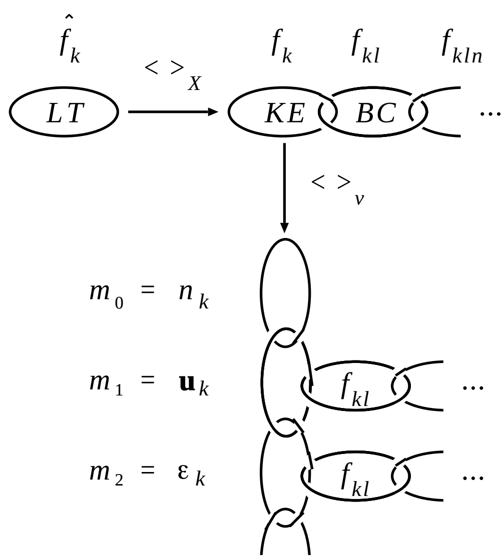


Figure 4.3: KE and BC are the kinetic equation and the equation for the correlation function. m_0, m_1 etc. are the chain of the equation for the moments.

For example, formula for the **mean collisional force** contains the unknown correlation function f_{kl}.

The last should be found from the correlation function Equation (the second link BC in Fig. 4.3).

The terms (4.37) and (4.38) in the energy conservation equation also depend on the unknown function f_{kl}.

It is also important that the transfer equations are unclosed in '**orthogonal**' direction:

Equation for the zeroth moment (the link m_0 in Fig. 4.3), density n_k, depends on the unknown first moment, the mean velocity $\mathbf{u}_k$, and so on.

This **process of generating equations** for the higher moments could be extended indefinitely depending solely on how many primary variables (n_k, $\mathbf{u}_k$, ε_k, ...) we are prepared to introduce.

Anyway we know now that

> the conservation laws for mass, momentum, and energy in the components of astrophysical plasma represent the **first three** links in the chain of equations for the distribution function moments.

It certainly would **not** be easy (if possible) to arrive to this fundamental conclusion

and

would be difficult to derive the conservation laws in the **form** of the transfer Equations (4.14), (4.17), and (4.23) in the way which is **typical** for the majority of textbooks: from simple specific knowledge to more general ones.

4.6 Equation of state and transfer coefficients

The **first three** transfer equations for a plasma component k would be closed with respect to the **three unknown** variables ρ_k, $\mathbf{u}_k$, and ε_k, if it were possible to express the other **unknown** quantities p_k, $\pi_{\alpha\beta}^{(k)}$, $q_\alpha^{(k)}$, etc. in terms of these three variables.

Thus, we have to know the **equation of state** and the so-called **transfer coefficients**.

How can we find them?

Formally, we should write **equations for higher moments** of the distribution function.

However these equations will not be closed either.

So, how shall we proceed?

According to the general principles of statistical physics,

> by virtue of collisions in a closed system of particles, any distribution function tends to assume the Maxwellian form.

The Maxwellian distribution is the kinetic equation solution for a stationary homogeneous gas in the absence of any mean force in the thermal equilibrium, i.e. for a gas in **thermodynamic equilibrium**.

Then spatial gradients and derivatives with respect to time are **zero**.

In fact they are always **nonzero**.

For this reason, the assumption of **full** thermodynamic equilibrium is replaced with the **local** thermodynamic equilibrium (LTE).

Moreover

> if the gradients and derivatives are **small**, then the real distribution function differs **little** from the local Maxwellian one, the difference being **proportional** to the small gradients or derivatives.

If we are interested in a process occurring in a time t, which is much greater than the collision time τ, and at a distance L, which is much larger than the mean free path λ,

$$t \gg \tau \,, \qquad L \gg \lambda \,, \tag{4.42}$$

then the distribution function $f_k(\mathbf{r}, \mathbf{v}, t)$ is a sum of the **local** Maxwellian distribution

$$f_k^{(0)}(\mathbf{r}, \mathbf{v}, t) = n_k(\mathbf{r}, t) \left[\frac{m_k}{2\pi \, k_{\mathrm{B}} T_k(\mathbf{r}, t)} \right]^{3/2} \times$$

$$\times \exp \left\{ - \frac{m_k \, |\, \mathbf{v} - \mathbf{u}_k(\mathbf{r}, t) \,|^{\,2}}{2 \, k_{\mathrm{B}} T_k(\mathbf{r}, t)} \right\} \tag{4.43}$$

and some **small** additional term $f_k^{(1)}(\mathbf{r}, \mathbf{v}, t)$.

Therefore

$$f_k(\mathbf{r}, \mathbf{v}, t) = f_k^{(0)}(\mathbf{r}, \mathbf{v}, t) + f_k^{(1)}(\mathbf{r}, \mathbf{v}, t) \,. \tag{4.44}$$

Since the function $f_k^{(0)}$ depends on t and $\mathbf{r}$, we find the derivatives $\partial f_k^{(0)} / \partial t$ and $\partial f_k^{(0)} / \partial r_\alpha$.

By using these derivatives, we substitute function (4.44) in the kinetic equation and linearly approximate the collisional integral by using one or another of the models introduced in Chapter 3; see also Exercise 4.5.

Then we seek the additional term $f_k^{(1)}$ in the **linear** approximation.

For example, in the case of the heat flux q_α, the flux q_α is chosen to be proportional to the temperature gradient.

Thus, in a fully ionized plasma without magnetic field, the **heat flux** in the electron component of plasma

$$q_{\rm e} = -\,\kappa_{\rm e}\,\nabla T_{\rm e}\,, \tag{4.45}$$

where

$$\kappa_{\rm e} \approx \frac{1.84 \times 10^{-5}}{\ln\Lambda}\, T_e^{5/2} \tag{4.46}$$

is the coefficient of electron **thermal conductivity**.

In the presence of **strong magnetic field** in astrophysical plasma, all the transfer coefficients become **highly anisotropic**.

Since the Maxwellian function and its derivatives are determined by the parameters n_k, $\mathbf{u}_k$, and T_k, the **transfer coefficients** are expressed in terms of the same quantities and magnetic field B, of course.

> This procedure makes it possible to close the set of transfer equations for astrophysical plasma

under the conditions (4.42).

*   *   *

The **first three** moment equations were extensively used in astrophysics, for example, in investigations of the **solar wind**.

They led to a significant understanding of phenomena such as **escape**, **acceleration** and **cooling**.

However, as more detailed observations become available, it appeared that the **collisionally dominated** models are **not** adequate for most physical states of the solar wind.

A higher order, closed set of equations for the **six moments** have been derived for multi-fluid, **moderately non-Maxwellian plasma** of the solar wind.

On this basis, the generalized expression for heat flux relates the flux to the temperature gradients, relative streaming velocity, thermal anisotropy, temperature differences of the components.

4.7 Gravitational systems

There is a big difference between astrophysical plasmas and astrophysical gravitational systems (Sect. 3.3).

The gravitational **attraction cannot be screened**.

A large-scale gravitational field **always exists** over a system because the neutrality condition (3.17) cannot be satisfied.

The large-scale gravitational field makes an overall thermodynamic equilibrium impossible.

Therefore

> those results of plasma astrophysics which explicitly depend upon the plasma being in thermodynamic equilibrium do **not** hold for gravitational systems.

For systems, like the **stars in a galaxy**, we may hope that the observed distribution function reflects something about the **initial conditions** rather than just the relaxation mechanism.

So galaxies may be providing us with clues on **how they were formed**.

* * *

If we consider processes on a spatial scale which is large enough to contain a **large number** of stars then one of the main requirements of the continuum mechanics is justified.

Anyway, several aspects of the structure of a galaxy can be understood in **hydrodynamic approximation**.

More often than never,

> hydrodynamics provides a **first level** description of an astrophysical phenomenon governed predominantly by the gravitational force.

For example, the early stages of star formation during which an **interstellar cloud** of low density collapses under the action of its own gravity can be modeled in the hydrodynamic approximation.

However, when we want to explain the difference between the angular momentum of the cloud and that of the born star, we have to include the effect of a **magnetic field**.

Chapter 5

The Generalized Ohm's Law in Plasma

The multi-fluid models of astrophysical plasma allow us to derive the **generalized** Ohm's law and to consider different physical approximations, including the **collisional** and **collisionless** plasma models.

5.1 The classic Ohm's law

The usual Ohm's law,

$$\mathbf{j} = \sigma \mathbf{E},$$

relates the current $\mathbf{j}$ to the electric field $\mathbf{E}$ in a **solid conductor in rest**.

As we know, the electric field in every equation of motion determines **acceleration** of particles rather than their velocity.

That is why, generally, such a simple relation as the classic Ohm's law does **not** exist.

Moreover, while considering astrophysical plasmas, it is necessary to take into account the presence of a **magnetic field** and the **motion** of a plasma as a whole or as a medium consisting of several moving components, their **compressibility**.

Recall the way of deriving the usual Ohm's law.

The current is determined by the relative motion of electrons and ions.

Let us assume that the ions do not move.

An equilibrium is set up between the electric field action and electrons-on-ions **friction**:

$$0 = -\, e\, n_{\rm e}\, E_{\alpha} + \, m_{\rm e} n_{\rm e}\, \nu_{\rm ei}\, (\, 0 - u_{{\rm e},\alpha}) \;,$$

resulting in Ohm's law

$$j_{\alpha} = -\, e\, n_{\rm e}\, u_{{\rm e},\alpha} = + \frac{e^2 n_{\rm e}}{m_{\rm e}\, \nu_{\rm ei}}\, E_{\alpha} = \sigma E_{\alpha} \,. \qquad (5.1)$$

Here

$$\sigma = \frac{e^2 n_{\rm e}}{m_{\rm e}\, \nu_{\rm ei}} \qquad (5.2)$$

is the **electric conductivity**.

In order to deduce the generalized Ohm's law for a plasma **with** magnetic field, we have to consider **at least two** equations of motion – for the electron and ion components.

5.2 Derivation of basic equations

Let us write the momentum equations for electrons and ions:

$$m_{\mathrm{e}} \frac{\partial}{\partial t} (n_{\mathrm{e}}\, u_{\mathrm{e},\alpha}) = - \frac{\partial\, \Pi_{\alpha\beta}^{(\mathrm{e})}}{\partial r_\beta} - e n_{\mathrm{e}} \left[\mathbf{E} + \frac{1}{c} (\, \mathbf{u}_{\mathrm{e}} \times \mathbf{B}\,) \right]_\alpha + \\ + m_{\mathrm{e}} n_{\mathrm{e}}\, \nu_{\mathrm{ei}} \, (u_{\mathrm{i},\alpha} - u_{\mathrm{e},\alpha}) \,, \tag{5.3}$$

$$m_{\mathrm{i}} \frac{\partial}{\partial t} (n_{\mathrm{i}}\, u_{\mathrm{i},\alpha}) = - \frac{\partial\, \Pi_{\alpha\beta}^{(\mathrm{i})}}{\partial r_\beta} + Z_{\mathrm{i}}\, e n_{\mathrm{i}} \left[\mathbf{E} + \frac{1}{c} (\, \mathbf{u}_{\mathrm{i}} \times \mathbf{B}\,) \right]_\alpha + \\ + m_{\mathrm{e}} n_{\mathrm{e}}\, \nu_{\mathrm{ei}} \, (u_{\mathrm{e},\alpha} - u_{\mathrm{i},\alpha}) \,. \tag{5.4}$$

Here the tensor of momentum flux

$$\Pi_{\alpha\beta}^{(\mathrm{e})} (\mathbf{r}, t) = m_{\mathrm{e}} n_{\mathrm{e}}\, u_{\mathrm{e},\alpha}\, u_{\mathrm{e},\beta} + p_{\alpha\beta}^{(\mathrm{e})} \tag{5.5}$$

and

$$\Pi_{\alpha\beta}^{(\mathrm{i})} (\mathbf{r}, t) = m_{\mathrm{i}} n_{\mathrm{i}}\, u_{\mathrm{i},\alpha}\, u_{\mathrm{i},\beta} + p_{\alpha\beta}^{(\mathrm{i})} \,. \tag{5.6}$$

The last term in (5.3) represents the mean momentum transferred, because of collisions, between electrons and ions.

It is equal, with opposite sign, to the last term in Equation (5.4).

We assume that there are just **two kinds** of particles, their total momentum remaining constant under the action of **elastic collisions**.

Now let us suppose that the ions are protons ($Z_{\rm i} = 1$), and electrical **neutrality** occurs:

$$n_{\rm i} = n_{\rm e} = n\,.$$

Let us multiply (5.3) by $-e/m_{\rm e}$ and add it to (5.4) multiplied by $e/m_{\rm i}$.

The result is

$$\frac{\partial}{\partial t}\left[\, en\,(u_{{\rm i},\alpha} - u_{{\rm e},\alpha})\,\right] = \left[\,\frac{e}{m_{\rm i}}\,F_{{\rm i},\alpha} - \frac{e}{m_{\rm e}}\,F_{{\rm e},\alpha}\right] +$$

$$+\,e^2 n\left(\frac{1}{m_{\rm e}} + \frac{1}{m_{\rm i}}\right)E_\alpha + \frac{e^2 n}{c}\left[\left(\frac{\mathbf{u}_{\rm e}}{m_{\rm e}} \times \mathbf{B}\right)_\alpha + \left(\frac{\mathbf{u}_{\rm i}}{m_{\rm i}} \times \mathbf{B}\right)_\alpha\right] -$$

$$-\,\nu_{\rm ei}\,en\left[(u_{{\rm i},\alpha} - u_{{\rm e},\alpha}) + \frac{m_{\rm e}}{m_{\rm i}}\,(u_{{\rm i},\alpha} - u_{{\rm e},\alpha})\right]. \qquad (5.7)$$

Here

$$F_{{\rm e},\alpha} = -\,\frac{\partial\,\Pi^{(\rm e)}_{\alpha\beta}}{\partial r_\beta} \quad \text{and} \quad F_{{\rm i},\alpha} = \frac{\partial\,\Pi^{(\rm i)}_{\alpha\beta}}{\partial r_\beta}\,. \qquad (5.8)$$

Let us introduce the velocity of the centre-of-mass system

$$\mathbf{u} = \frac{m_{\rm i}\,\mathbf{u}_{\rm i} + m_{\rm e}\,\mathbf{u}_{\rm e}}{m_{\rm i} + m_{\rm e}}\,.$$

Since $m_{\rm i} \gg m_{\rm e}$,

$$\mathbf{u} = \mathbf{u}_{\rm i} + \frac{m_{\rm e}}{m_{\rm i}}\,\mathbf{u}_{\rm e} \approx \mathbf{u}_{\rm i}\,. \qquad (5.9)$$

On treating Equation (5.7), we neglect the small terms of the order of the ratio $m_{\mathrm{e}}/m_{\mathrm{i}}$.

We obtain the equation for the current

$$\mathbf{j} = en\,(\mathbf{u}_{\mathrm{i}} - \mathbf{u}_{\mathrm{e}})$$

in the system of coordinates (5.9).

This equation is

$$\frac{\partial \mathbf{j}'}{\partial t} = \frac{e^2 n}{m_{\mathrm{e}}} \left[\mathbf{E} + \frac{1}{c} (\, \mathbf{u} \times \mathbf{B} \,) \right] - \frac{e}{m_{\mathrm{e}} c} (\, \mathbf{j}' \times \mathbf{B} \,) -$$

$$- \nu_{\mathrm{ei}} \mathbf{j}' + \frac{e}{m_{\mathrm{i}}} \mathbf{F}_{\mathrm{i}} - \frac{e}{m_{\mathrm{e}}} \mathbf{F}_{\mathrm{e}} \,. \qquad (5.10)$$

The prime designates the current in the system of moving plasma, i.e. in the rest-frame of the plasma.

Let $\mathbf{E}_{\mathrm{u}}$ denote the electric field in this frame of reference, i.e.

$$\mathbf{E}_{\mathrm{u}} = \mathbf{E} + \frac{1}{c} \mathbf{u} \times \mathbf{B} \,. \qquad (5.11)$$

Now we divide Equation (5.10) by ν_{ei} and represent it in the form

$$\mathbf{j}' = \frac{e^2 n}{m_{\mathrm{e}} \nu_{\mathrm{ei}}} \mathbf{E}_{\mathrm{u}} - \frac{\omega_{\mathrm{B}}^{(\mathrm{e})}}{\nu_{\mathrm{ei}}} \mathbf{j}' \times \mathbf{n} -$$

$$- \frac{1}{\nu_{\mathrm{ei}}} \frac{\partial \mathbf{j}'}{\partial t} + \frac{1}{\nu_{\mathrm{ei}}} \left(\frac{e}{m_{\mathrm{i}}} \mathbf{F}_{\mathrm{i}} - \frac{e}{m_{\mathrm{e}}} \mathbf{F}_{\mathrm{e}} \right) . \qquad (5.12)$$

Here

$$\mathbf{n} = \mathbf{B}/B$$

and

$$\omega_{\mathrm{B}}^{(\mathrm{e})} = \frac{eB}{m_{\mathrm{e}}c}$$

is the electron gyro-frequency.

Thus we have derived a **differential** equation for the current $\mathbf{j}'$.

The third and the fourth terms on the right do **not** depend of magnetic field.

Let us replace them by some **effective** electric field

$$\sigma \mathbf{E}_{\mathrm{eff}} = -\frac{1}{\nu_{\mathrm{ei}}} \frac{\partial \mathbf{j}'}{\partial t} + \frac{e}{\nu_{\mathrm{ei}}} \left(\frac{1}{m_{\mathrm{i}}} \mathbf{F}_{\mathrm{i}} - \frac{1}{m_{\mathrm{e}}} \mathbf{F}_{\mathrm{e}} \right), \tag{5.13}$$

where

$$\boxed{\sigma = \frac{e^2 n}{m_{\mathrm{e}}\, \nu_{\mathrm{ei}}}} \tag{5.14}$$

is the **plasma conductivity** in the absence of magnetic field.

Combine the fields (5.11) and (5.13),

$$\mathbf{E}' = \mathbf{E}_{\mathrm{u}} + \mathbf{E}_{\mathrm{eff}}\,, \tag{5.15}$$

in order to rewrite (5.12) in the form

$$\mathbf{j}' = \sigma \mathbf{E}' - \frac{\omega_{\rm B}^{\rm (e)}}{\nu_{\rm ei}} \, \mathbf{j}' \times \mathbf{n} \, . \tag{5.16}$$

We shall consider (5.16) as an **algebraic** equation in $\mathbf{j}'$, neglecting the $\partial \, \mathbf{j}'/\partial t$ dependence of the field (5.13).

Note, however, that

> the term $\partial \, \mathbf{j}'/\partial t$ is by **no** means small in the problem of the particle **acceleration** by a strong electric field in astrophysical plasma.

Collisionless reconnection is the phenomenon in which **particle inertia** of the current replaces classical resistivity in allowing **fast** reconnection to occur.

5.3 The general solution

Let us find the solution to Equation (5.16) as a sum

$$\boxed{\mathbf{j}' = \sigma_{\parallel} \, \mathbf{E}'_{\parallel} + \sigma_{\perp} \, \mathbf{E}'_{\perp} + \sigma_{\rm H} \, \mathbf{n} \times \mathbf{E}'_{\perp} \, .} \tag{5.17}$$

Substituting (5.17) in (5.16) gives

$$\sigma_{\parallel} = \sigma = \frac{e^2 n}{m_{\rm e} \nu_{\rm ei}} \; , \tag{5.18}$$

$$\sigma_{\perp} = \sigma \, \frac{1}{1 + \left(\omega_{\rm B}^{\rm (e)} \tau_{\rm ei} \right)^2} \; , \tag{5.19}$$

$$\sigma_{\mathrm{H}} = \sigma \, \frac{\omega_{\mathrm{B}}^{(\mathrm{e})} \tau_{\mathrm{ei}}}{1 + \left(\omega_{\mathrm{B}}^{(\mathrm{e})} \tau_{\mathrm{ei}}\right)^2} \; . \tag{5.20}$$

Formula (5.17) is called the **generalized** Ohm's law.

A magnetic field in a plasma not only changes the magnitude of the conductivity, but the **form** of Ohm's law as well:

the electric field and the resulting current are **not** parallel, since

$$\sigma_{\perp} \neq \sigma_{\parallel} \, .$$

Thus the conductivity of a plasma in a magnetic field is **anisotropic**.

Moreover the current component $\mathbf{j}'_{\mathrm{H}}$ is perpendicular to **both** the magnetic **and** electric fields.

This component is the so-called **Hall current** (Fig. 5.1).

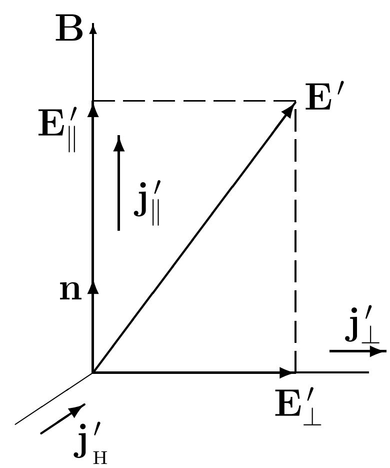


Figure 5.1: The direct ($\mathbf{j}'_{\parallel}$ and $\mathbf{j}'_{\perp}$) and Hall's ($\mathbf{j}'_{\mathrm{H}}$) currents in a plasma with electric ($\mathbf{E}'$) and magnetic ($\mathbf{B}$) fields.

5.4 The conductivity of magnetized plasma

5.4.1 Two limiting cases

The magnetic-field influence on the conductivity $\sigma_\perp$ and on the Hall conductivity σ_{H} is determined by the parameter

$$\omega_{\text{B}}^{(e)} \tau_{ei} \, .$$

This is the **turning angle** of an electron on the Larmor circle in the intercollisional time.

Let us consider **two limiting cases**.

(a) The turning angle be small:

$$\omega_{\text{B}}^{(e)} \tau_{ei} \ll 1 \, . \tag{5.21}$$

Obviously this corresponds to the **weak** magnetic field or **dense cool** plasma, so that the electric current is scarcely affected by the magnetic field:

$$\sigma_\perp \approx \sigma_\parallel = \sigma \, , \qquad \frac{\sigma_{\text{H}}}{\sigma} \approx \omega_{\text{B}}^{(e)} \tau_{ei} \ll 1 \, . \tag{5.22}$$

Thus the usual Ohm's law with **isotropic** conductivity holds.

(b) The opposite case, when the electrons **spiral freely** between rare collisions of electrons with ions:

$$\omega_{\text{B}}^{(e)} \tau_{ei} \gg 1 \, , \tag{5.23}$$

corresponds to the **strong** magnetic field and **hot rarefied** plasma.

This plasma is termed the **magnetized** one.

It is frequently encountered under astrophysical conditions.

In this case

$$\sigma_{\,\parallel} = \sigma \;\approx \left(\omega_{_{\rm B}}^{\rm (e)} \tau_{\rm ei}\right) \sigma_{_{\rm H}} \approx \left(\omega_{_{\rm B}}^{\rm (e)} \tau_{\rm ei}\right)^2 \sigma_{\perp} \, . \tag{5.24}$$

Hence in a magnetized plasma, for example in the solar corona

$$\boxed{\sigma_{\parallel} \gg \sigma_{_{\rm H}} \gg \sigma_{\perp} \, .} \tag{5.25}$$

In other words,

> the impact of the magnetic field on the direct current is especially strong for the component resulting from the electric field $\mathbf{E}'_{\perp}$.

The current in the $\mathbf{E}'_{\perp}$ direction is **considerably weaker** than it would be in the absence of a magnetic field.

Why?

5.4.2 The physical interpretation

The physical mechanism of the perpendicular current $\mathbf{j}'_{\perp}$ is illustrated by Fig. 5.2.

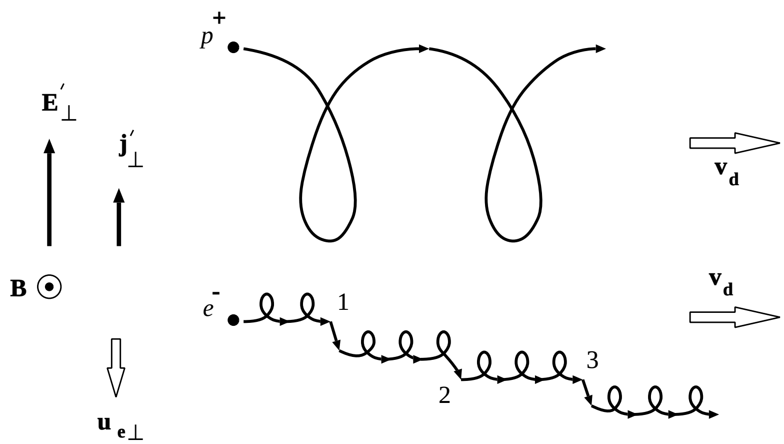


Figure 5.2: Initiation of the current in the direction of the perpendicular field $\mathbf{E}'_{\perp}$ as the result of rare collisions (1, 2, 3, ...).

The **primary effect** of the electric field $\mathbf{E}'_{\perp}$ in the presence of the magnetic field $\mathbf{B}$ is not the current in the direction $\mathbf{E}'_{\perp}$, but rather the electric **drift** in the direction perpendicular to both $\mathbf{B}$ and $\mathbf{E}'_{\perp}$.

The electric drift velocity is independent of the particle's mass and charge.

The electric drift of electrons and ions generates the motion of the plasma as a whole with the velocity

$$\mathbf{v} = \mathbf{v}_{\mathrm{d}} = c\,\frac{\mathbf{E}\times\mathbf{B}}{B^{2}}\,. \qquad (5.26)$$

This would be the case if there were **no collisions** at all.

Collisions, even the rare ones, disturb the Larmor motion, leading to a displacement of the ions (not shown in

Fig. 5.2) along the field $\mathbf{E}'_{\perp}$, and the electrons in the opposite direction (Fig. 5.2).

The **small** electric current $\mathbf{j}'_{\perp}$ appears in the direction $\mathbf{E}'_{\perp}$.

To ensure the current across the magnetic field, the **electric field** is necessary, i.e. the electric field component perpendicular to both the current $\mathbf{j}'_{\perp}$ and the field $\mathbf{B}$.

> The **Hall electric field** balances the Lorentz force acting on the carriers of the perpendicular electric current in a rarely collisional plasma due to the presence of a magnetic field,

i.e. the force

$$\mathbf{F}\,(\,\mathbf{j}'_{\perp}) = \frac{e\,n}{c}\,\mathbf{u}_{\mathrm{i}\perp} \times \mathbf{B} - \frac{e\,n}{c}\,\mathbf{u}_{\mathrm{e}\perp} \times \mathbf{B} =$$

$$= \frac{1}{c}\,e\,n\,(\mathbf{u}_{\mathrm{i}\perp} - \mathbf{u}_{\mathrm{e}\perp}) \times \mathbf{B} \tag{5.27}$$

Hence the magnitude of the Hall electric field is

$$\mathbf{E}'_{\mathrm{H}} = \frac{1}{en\,c}\,\mathbf{j}'_{\perp} \times \mathbf{B}\,. \tag{5.28}$$

The Hall electric field in astrophysical plasma is frequently set up automatically, as a consequence of **small charge separation** within the limits of quasi-neutrality.

In a **fully-ionized rarely-collisional** plasma, the tendency for a particle to spiral round the magnetic field lines insures the great reduction in the transversal conductivity.

However, since the dissipation of the energy of the electric current into Joule heat,

$$\mathbf{j}'\,\mathbf{E}' \,,$$

is due solely to collisions between particles (if the particle acceleration can be neglected), the reduced conductivity does **not** lead to increased dissipation.

On the other hand, the Hall electric field and **Hall current** can significantly modify conditions of magnetic **reconnection**.

Compared with ordinary resistive MHD, the Hall MHD reconnection is distinguished by **qualitatively different** magnetic field distributions, electron and ion signatures in **reconnecting current layers**.

Although the Hall effect itself is **nondissipative**,

$$\mathbf{j}'_{\mathrm{H}}\,\mathbf{E}' = 0 \,, \tag{5.29}$$

it can lead to dissipation through a turbulent "**Hall cascade**", magnetic energy cascading from large to small scales, where it dissipates by ohmic decay.

The Hall effect can dominate ohmic decay of currents in the crust of **neutron stars** and therefore can determine evolution of their magnetic field.

In an initial poloidal **dipole** field, the toroidal currents "twist" the field.

The resulting poloidal currents then generate a **quadrupole** poloidal field.

5.5 Currents and charges in plasma

5.5.1 Collisional and collisionless plasmas

Let us point out another property of the generalized Ohm's law.

Under laboratory conditions, as a rule, one cannot neglect the **gradient forces**.

On the contrary, these forces usually play **no** part in astrophysical plasma.

We shall often ignore them.

However this simplification may be **not** justified in **reconnecting current layers**, shock waves and other **discontinuities**.

Let us also restrict our consideration to very **slow** (say **hydrodynamic**) motions of plasma.

These motions are supposed to be so slow that the following **three conditions** are fulfilled.

(A) It is supposed that

$$\omega = \frac{1}{\tau} \ll \nu_{\rm ei} \quad \text{or} \quad \nu_{\rm ei}\, \tau \gg 1\,, \tag{5.30}$$

where τ is a characteristic time of the plasma motions.

Thus

departures of actual distribution functions for electrons and ions from the Maxwellian distribution are small.

This allows us to handle the transfer phenomena in linear approximation.

Moreover, if a **single-fluid** model makes a sense, the electrons and ions could have comparable temperatures, **ideally**, the same one T which is the temperature of the plasma as a whole:

$$T_{\mathrm{e}} = T_{\mathrm{p}} = T\,.$$

(B) We neglect the electron inertia in comparison with that of the ions and make use of (5.9).

This condition is usually written in the form

$$\omega \ll \omega_{\mathrm{B}}^{(\mathrm{i})} = \frac{eB}{m_{\mathrm{i}}c}\,. \tag{5.31}$$

Thus

the plasma motions have to be so slow that their frequency is smaller than the **lowest gyro-frequency** of the particles.

Recall that the gyro-frequency of ions

$$\omega_{\mathrm{B}}^{(\mathrm{i})} \ll \omega_{\mathrm{B}}^{(\mathrm{e})}\,.$$

(C) The third condition

$$\omega_{_{\rm B}}^{(\rm e)} \tau_{\rm ei} \ll 1\,. \tag{5.32}$$

Hence we can use the **isotropic** conductivity σ.

The generalized Ohm's law assumes the form which is specific to the ordinary **magnetohydrodynamics** (MHD):

$$\mathbf{j}' = \sigma \left(\mathbf{E} + \frac{1}{c}\, \mathbf{u} \times \mathbf{B} \right). \tag{5.33}$$

The MHD approximation is the subject of the next chapter.

Numerous applications of MHD to astrophysical plasma should be discussed in the remainder of the lectures.

*   *   *

In the opposite case, when the parameter

$$\omega_{_{\rm B}}^{(\rm e)}\, \tau_{\rm ei} \gg 1\,,$$

charged **particles revolve around** magnetic field lines, and a typical particle may spend a considerable time in a region of a size of the order of the gyroradius.

Hence, if the length scale of a phenomenon is much larger than the gyroradius, we may expect the **hydrodynamic-type** models to work.

It appears that, even when the parameter

$$\omega_{_{\rm B}}^{(\rm e)}\, \tau_{\rm ei} \to \infty\,,$$

(like in the solar corona) and collisions are negligible, the 2D **quasi-hydrodynamic** description of plasma, the **Chew-Goldberger-Low**(CGL) approximation is quite useful.

This is because

> a strong magnetic field makes a plasma, even a collisionless one, more '**interconnected**', more hydrodynamic in the directions perpendicular to the magnetic field.

As for the motion of particles **along** the magnetic field, some important **kinetic features** still are significant.

Chew et al.: "A strictly hydrodynamic approach to the problem is appropriate only when some special circumstance suppresses the effects of **pressure transport** along the magnetic lines".

There is ample experimental evidence that strong magnetic fields do make astrophysical plasmas behave like **hydrodynamic charged fluids**.

This does not mean, of course, that there are no **pure kinetic phenomena** in such plasmas.

There are many of them indeed.

The most interesting of them is **magnetic reconnection** in the solar corona and solar wind.

5.5.2 Volume charge and quasi-neutrality

While deriving the generalized Ohm's law, the **exact** charge neutrality of plasma was assumed:

$$\sum_i Z_\mathrm{i} n_\mathrm{i} = n_\mathrm{e}\,,$$

i.e. the absolute absence of the **volume charge** in plasma:

$$\rho^{\,\mathrm{q}} = 0\,.$$

However there is **no** need for such a strong restriction.

It is sufficient to require **quasi-neutrality**, i.e.

$$\left(\sum_i Z_\mathrm{i} n_\mathrm{i} - n_\mathrm{e}\right) n_\mathrm{e}^{-1} \ll 1\,.$$

So

> the volume charge density has to be small in comparison to the plasma density.

Once the **volume charge** density

$$\rho^{\,\mathrm{q}} = e\,\left(\sum_i Z_\mathrm{i} n_\mathrm{i} - n_\mathrm{e}\right) \neq 0\,,$$

yet another term must be taken into account in the Ohm's law:

$$\mathbf{j}_\mathrm{u}^{\,\mathrm{q}} = \rho^{\,\mathrm{q}}\,\mathbf{u}\,. \tag{5.34}$$

This is the so-called **convective** current.

It must be added to the **conductive** current (5.17).

The volume charge, the associated electric force $\rho^{\,\mathrm{q}}\,\mathbf{E}$ and the convective current $\rho^{\,\mathrm{q}}\,\mathbf{u}$ are of great importance in electrodynamics of **relativistic** objects such as **black holes** and **pulsars**.

Charge-separated plasmas originate in magnetospheres of pulsars and **rotating** black holes, e.g., a **supermassive** black hole in active galactic nuclei (AGN).

A **strong electric field** appears along the magnetic field lines.

The parallel electric field accelerates migratory electrons and/or positrons to **ultra-relativistic energies**.

* * *

Volume charge can be evaluated in the following manner.

From Maxwell's equation

$$\operatorname{div}\mathbf{E} = 4\pi\rho^{\,\mathrm{q}}$$

we estimate

$$\rho^{\,\mathrm{q}} \approx \frac{E}{4\pi L}\,. \tag{5.35}$$

On the other hand, the equation of plasma motion yields

$$e n_{\mathrm{e}} E \approx \frac{p}{L} \approx \frac{n_{\mathrm{e}} k_{\mathrm{B}} T}{L}\,,$$

so that

$$E \approx \frac{k_{\mathrm{B}} T}{eL} \,. \tag{5.36}$$

On substituting (5.36) in (5.35), we find

$$\frac{\rho^{\mathrm{q}}}{en_{\mathrm{e}}} \approx \frac{k_{\mathrm{B}} T}{eL} \, \frac{1}{4\pi L} \, \frac{1}{en_{\mathrm{e}}} = \frac{1}{L^2} \left(\frac{k_{\mathrm{B}} T}{4\pi e^2 \, n_{\mathrm{e}}} \right)$$

or

$$\boxed{\frac{\rho^{\mathrm{q}}}{en_{\mathrm{e}}} \approx \frac{r_{\mathrm{DH}}^{\,2}}{L^2} \,.} \tag{5.37}$$

Since the usual **concept of plasma** implies that the Debye radius

$$r_{\mathrm{DH}} \ll L \,, \tag{5.38}$$

the volume charge density is small in comparison with the plasma density.

When we consider phenomena with a length scale L much larger than the Debye radius r_{DH} and a time scale τ much larger than the inverse the plasma frequency, the **charge separation can be neglected**.

5.6 Practice: Exercises and Answers

Exercise 5.1 Consider a plasma system with given distributions of magnetic and velocity fields.

Is it possible to use Equation (5.12) in order to estimate the growth rate of electric current and, as a consequence, of magnetic field in such a system, for example, a protostar?

Exercise 5.2 Evaluate the characteristic value of the parallel conductivity (5.18) in the **solar corona**.

Answer. It follows from formula (5.18) that

$$\sigma_{\parallel} = \frac{e^2 n}{m_{\rm e}} \, \tau_{\rm ei} \sim 10^{16} - 10^{17} \, , \;\; {\rm s}^{-1} \, . \tag{5.39}$$

Exercise 5.3 Estimate the parameter $\omega_{\rm B}^{(e)} \tau_{\rm ei}$ in the corona above a sunspot.

Answer. Just above a large sunspot the field strength can be as high as $B \approx 3000$ G .

With $\tau_{\rm ep} \approx 0.1$ s, we obtain

$$\omega_{\rm B}^{(e)} \tau_{\rm ei} \sim 10^{10} \, {\rm rad} \gg 1 \, .$$

So, for anisotropic conductivity in the solar corona, the approximate formulae (**??**) can be well used.

Chapter 6

Single-Fluid Models for Astrophysical Plasma

Single-fluid models are the simplest but sufficient approximation to describe many **large-scale low-frequency phenomena** in astrophysical plasma: motions driven by strong magnetic fields, accretion disks, and relativistic jets.

6.1 Derivation of the single-fluid equations

6.1.1 The continuity equation

In order to consider a plasma as a **single** medium, we have to sum each of the three transfer equations over all kinds of particles.

Let us start from the continuity equation

$$\frac{\partial n_k}{\partial t} + \frac{\partial}{\partial r_\alpha}\, n_k\, u_{k,\alpha} = 0\,.$$

With allowance for the definition of the plasma mass density ρ, we have

$$\frac{\partial \rho}{\partial t} + \operatorname{div}\left(\sum_k \rho_k \mathbf{u}_k\right) = 0\,. \tag{6.1}$$

The **mean** velocities of motion for all kinds of particles are supposed to be equal to the plasma **hydrodynamic** velocity:

$$\mathbf{u}_1(\mathbf{r},t) = \mathbf{u}_2(\mathbf{r},t) = \cdots = \mathbf{u}(\mathbf{r},t)\,, \tag{6.2}$$

as a result of action of the mean collisional force.

However this is not a general case.

In general, the mean velocities are **not** the same, but a frame of reference can be chosen in which

$$\rho\,\mathbf{u} = \sum_k \rho_k \mathbf{u}_k\,. \tag{6.3}$$

Then from (6.1) and (6.3) we obtain the usual **continuity equation**

$$\boxed{\frac{\partial \rho}{\partial t} + \operatorname{div}\rho\,\mathbf{u} = 0\,.} \tag{6.4}$$

We shall consider both cases.

6.1.2 The momentum conservation law

In the same way, we handle the momentum equation

$$\rho_k \, \frac{d^{(k)} u_{k,\alpha}}{dt} = - \, \frac{\partial}{\partial r_\beta} \, p^{(k)}_{\alpha\beta} + \rho^{\,\mathrm{q}}_k \, E_\alpha + \frac{1}{c} \, (\, \mathbf{j}^{\,\mathrm{q}}_k \times \mathbf{B} \,)_\alpha \, +$$

$$+ \, \langle \, F^{(\mathrm{c})}_{k,\alpha} \, (\mathbf{r}, t) \, \rangle_{\, v} \, .$$

On summing over all kinds of particles, we obtain the equation

$$\rho \, \frac{d \, u_\alpha}{dt} = - \, \frac{\partial}{\partial r_\beta} \, p_{\alpha\beta} + \rho^{\,\mathrm{q}} \, E_\alpha + \frac{1}{c} \, (\, \mathbf{j} \times \mathbf{B} \,)_\alpha \, +$$

$$+ \sum_k \langle \, F^{(\mathrm{c})}_{k,\alpha} \, (\mathbf{r}, t) \, \rangle_v \, . \tag{6.5}$$

Here the **volume charge** is

$$\rho^{\,\mathrm{q}} = \sum_k \, n_k e_k = \frac{1}{4\pi} \, \mathrm{div} \, \mathbf{E} \, , \tag{6.6}$$

and the **electric current** is

$$\mathbf{j} = \sum_k \, n_k e_k \, \mathbf{u}_k = \frac{c}{4\pi} \, \mathrm{rot} \, \mathbf{B} - \frac{1}{4\pi} \, \frac{\partial \mathbf{E}}{\partial t} \, . \tag{6.7}$$

The electric and magnetic fields, $\mathbf{E}$ and $\mathbf{B}$, are averaged fields associated with the **total** electric charge density $\rho^{\,\mathrm{q}}$ and the **total** current $\mathbf{j}$.

They satisfy the **macroscopic** Maxwell equations.

Since **elastic** collisions do not change the total momentum,

$$\sum_k \langle F^{(c)}_{k,\alpha} (\mathbf{r}, t) \rangle_v = 0 \,. \tag{6.8}$$

On substituting (6.6)–(6.8) in Equation (6.5), the latter gives the **momentum conservation law**

$$\boxed{\rho \frac{d\, u_\alpha}{dt} = - \frac{\partial}{\partial r_\beta} p_{\alpha\beta} + F_\alpha(\mathbf{E}, \mathbf{B}) \,.} \tag{6.9}$$

Here the **electromagnetic force** is written in terms of the electric and magnetic fields:

$$F_\alpha(\mathbf{E}, \mathbf{B}) = - \frac{\partial}{\partial t} \frac{(\, \mathbf{E} \times \mathbf{B} \,)_\alpha}{4 \pi c} - \frac{\partial}{\partial r_\beta} \mathrm{M}_{\alpha\beta} \,. \tag{6.10}$$

The tensor

$$\mathrm{M}_{\alpha\beta} = \frac{1}{4\pi} \left[- E_\alpha E_\beta - B_\alpha B_\beta + \frac{1}{2} \, \delta_{\alpha\beta} \, (E^2 + B^2 \,) \right] \tag{6.11}$$

is the **Maxwellian tensor** of stresses.

The divergent form of the **momentum conservation law** is

$$\boxed{\frac{\partial}{\partial t}\left[\rho\, u_\alpha + \frac{(\,\mathbf{E}\times\mathbf{B}\,)_\alpha}{4\pi c}\right] + \frac{\partial}{\partial r_\beta}\,(\,\Pi_{\alpha\beta} + \mathrm{M}_{\alpha\beta}\,)\; = 0\,.} \tag{6.12}$$

The operator $\partial/\partial t$ acts on two terms:

$\rho\,\mathbf{u}$ is the momentum of the plasma in a unit volume,

$\mathbf{E}\times\mathbf{B}/4\pi c$ is the momentum of the electromagnetic field.

The divergency operator $\partial/\partial r_\alpha$ acts on

$$\Pi_{\alpha\beta} = p_{\alpha\beta} + \rho\, u_\alpha u_\beta\,, \tag{6.13}$$

which is the **momentum flux** tensor

$$\Pi_{\alpha\beta} = \sum_k \Pi_{\alpha\beta}^{(k)}\,, \tag{6.14}$$

see definition (4.10).

Thus the **pressure tensor**

$$p_{\alpha\beta} = p\,\delta_{\alpha\beta} + \pi_{\alpha\beta}\,, \tag{6.15}$$

where

$$p = \sum_k p_k$$

is the total plasma pressure, the sum of **partial pressures**, and

$$\pi_{\alpha\beta} = \sum_k \pi^{(k)}_{\alpha\beta} \tag{6.16}$$

is the **viscous stress** tensor which allows for the transport of momentum from one layer of the plasma flow to the other layers so that relative motions inside the plasma are damped out.

The momentum conservation law (6.12) is applied for a wide range of conditions in plasmas like **fluid relativistic flows**, for example, astrophysical jets.

The assumption that the astrophysical plasma behaves as a continuum medium is **excellent** in the cases in which we are often interested:

> the Debye length and the Larmor radii are much smaller than the plasma flow scales.

On the other hand, going from the multi-fluid description to a single-fluid model is a **serious damage** because we loose an information not only on the **small-scale** dynamics of the electrons and ions but also on the **high-frequency** processes in plasma.

> The single-fluid equations describe well the **low-frequency large-scale** behavior of plasma in astrophysical conditions.

6.1.3 The energy conservation law

In a similar manner as above, the energy conservation law is derived.

We sum Equation (4.23) over k and then substitute in the resulting equation the **total** electric charge (6.6) and the **total** electric current (6.7) expressed in terms of the electric field **E** and magnetic field **B**.

The following divergent form of the energy conservation law is obtained:

$$\frac{\partial}{\partial t}\left(\frac{\rho u^2}{2} + \rho\,\varepsilon + \frac{E^2 + B^2}{8\pi}\right) + \\ + \frac{\partial}{\partial r_\alpha}\left[\rho\, u_\alpha\left(\frac{u^2}{2} + w\right) + \frac{c}{4\pi}\,(\,\mathbf{E}\times\mathbf{B}\,)_\alpha + \pi_{\alpha\beta}\,u_\beta + \right. \\ \left. + q_\alpha\,\right] = \left(u_\alpha F_\alpha^{(\mathrm{c})}\right)_{ff} + \mathcal{L}^{(rad)}\,(\mathbf{r}, t)\,. \qquad (6.17)$$

On the left-hand side of this equation, an additional term has appeared:

the operator $\partial/\partial t$ acts on the energy density of the electromagnetic field

$$W = \frac{E^2 + B^2}{8\pi}\,. \qquad (6.18)$$

The divergency operator $\partial/\partial r_\alpha$ acts on the **Poynting vector**, the electromagnetic energy flux

$$\mathbf{G} = \frac{c}{4\pi} \, [\, \mathbf{E} \times \mathbf{B} \,] \, . \tag{6.19}$$

The right-hand side of Equation (6.17) contains the total work of **friction forces** in unit time on unit volume

$$\left(u_\alpha F_\alpha^{(\mathrm{c})} \right)_{ff} = \sum_k \left(F_{k,\alpha}^{(\mathrm{c})} \, u_{k,\alpha} \right) =$$

$$= \sum_k u_{k,\alpha} \int_{\mathbf{v}} m_k \, v'_\alpha \left(\frac{\partial f_k}{\partial t} \right)_{\mathrm{c}} d^{\,3}\mathbf{v} \, . \tag{6.20}$$

This work related to the **relative motion** of the plasma components is not zero.

Recall that we consider general case (6.3).

By contrast, the **total heat release** under elastic collisions between particles of different kinds is

$$\sum_k Q_k^{(c)} \, (\mathbf{r}, t) = \sum_k \int_{\mathbf{v}} \frac{m_k \, (v')^2}{2} \left(\frac{\partial f_k}{\partial t} \right)_{\mathrm{c}} d^{\,3}\mathbf{v} = 0 \, . \tag{6.21}$$

> Elastic collisions in a plasma conserve both the total momentum and the total energy.

If we accept condition (6.2) then the collisional heating (6.20) by friction force is also equal to zero.

In this limit, there is not any term which contains the collisional integral.

Elastic collisions have done a **good job**.

Inelastic collisions are important in radiative cooling and heating.

In optically thin plasma with collisional excitations of ions, the power of radiation from a unit volume of plasma is proportional to the square of plasma density $n\,(\mathrm{cm}^{-3})$:

$$\mathcal{L}^{(rad)} \simeq -\,n^2\,q(T)\,. \tag{6.22}$$

The function $q(T)$ is called the **radiative loss function**.

It depends strongly on the temperature T but weakly on the plasma density n (Fig. 6.1).

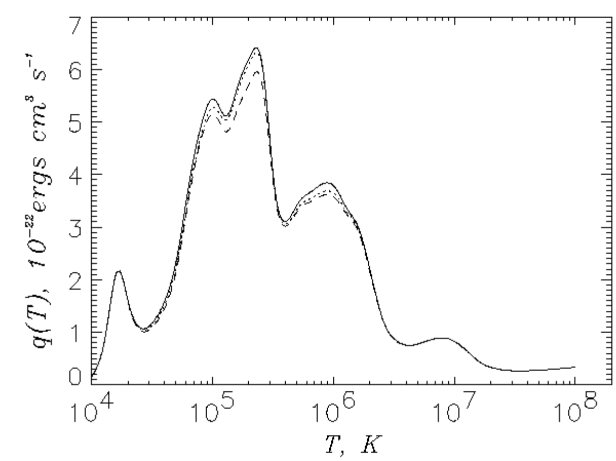


Figure 6.1: Radiative loss function vs. temperature at fixed values of plasma density: $10^9\,\mathrm{cm}^{-3}$ (solid curve), $10^{10}\,\mathrm{cm}^{-3}$ (dotted curve), $10^{11}\,\mathrm{cm}^{-3}$ (dashed curve).

6.2 Basic assumptions and the MHD equations

6.2.1 Old simplifying assumptions

As we saw above, the transfer equations determines the behavior of different kinds of particles in a plasma once **two conditions** are complied with:

(a) many collisions occur in a characteristic time τ of a phenomenon under consideration:

$$\tau \gg \tau_c \,, \tag{6.23}$$

(b) the mean free path λ_c is significantly smaller than the distance L, over which macroscopic quantities change considerably:

$$L \gg \lambda_c \,. \tag{6.24}$$

Once these conditions are satisfied, we can close the set of **transfer** equations, as was discussed in Sect. 4.6.

While considering the generalized Ohm's law, other **three assumptions** have been made.

The first condition can be written in the form

$$\tau \gg \tau_{\rm ei} \,, \tag{6.25}$$

where $\tau_{\rm ei}$ is the electron-ion collisional time, the **longest** collisional relaxation time.

Thus the electrons and ions have comparable temperatures, ideally, **the same temperature** T.

Second, we neglect the **electron inertia** in comparison with that of the ions.

This condition is usually written as

$$\tau \gg \left(\omega_{\rm B}^{(\rm i)}\right)^{-1}, \quad \text{where} \quad \omega_{\rm B}^{(\rm i)} = \frac{eB}{m_{\rm i}\, c}\,. \tag{6.26}$$

Thus the plasma motions have to be so slow that their frequency $\omega = 1/\tau$ is smaller than the lowest gyro-frequency of the particles.

The third condition,

$$\omega_{\rm B}^{(\rm e)} \tau_{\rm ei} \ll 1\,, \tag{6.27}$$

is necessary to write down Ohm's law in the form

$$\mathbf{j} = \sigma \left(\mathbf{E} + \frac{1}{c}\, \mathbf{v} \times \mathbf{B} \right) + \rho^{\rm q}\, \mathbf{v}\,. \tag{6.28}$$

Here $\mathbf{v}$ is the velocity of plasma, $\mathbf{E}$ and $\mathbf{B}$ are the electric and magnetic fields in the 'laboratory' system of coordinates, where we measure the velocity $\mathbf{v}$.

Accordingly, the field

$$\mathbf{E}_{\rm v} = \mathbf{E} + \frac{1}{c}\, \mathbf{v} \times \mathbf{B} \tag{6.29}$$

is the electric field in a frame of reference related to the plasma.

Complementary to the restriction (6.24) on the characteristic length L, we have to add the condition

$$L \gg r_{_{\rm DH}} \, , \tag{6.30}$$

where $r_{_{\rm DH}}$ is the Debye-Hückel radius.

Then the volume charge $\rho^{\,\rm q}$ is small in comparison with the plasma density ρ.

Under the conditions listed above, we use the **general** hydrodynamic-type equations: the conservation laws for mass (6.4), momentum (6.5) and energy (6.17).

> The **general** hydrodynamic-type equations have a much wider area of applicability in astrophysics than the equations of **ordinary** MHD derived below.

The latter will be **simpler** than the equations derived above.

Therefore **additional simplifying assumptions** are necessary.

Let us introduce them.

6.2.2 New simplifying assumptions

First assumption:

the conductivity σ is large, the electromagnetic processes being not very fast.

Then, in the Maxwell's equation

$$\operatorname{rot}\mathbf{B} = \frac{4\pi}{c}\,\mathbf{j} + \frac{1}{c}\,\frac{\partial \mathbf{E}}{\partial t}\,,$$

we ignore the **displacement** current in comparison to the **conductive** one.

The corresponding condition is found by evaluating the currents as follows

$$\frac{1}{c}\,\frac{E}{\tau} \ll \frac{4\pi}{c}\,j \quad \text{or} \quad \omega E \ll 4\pi\sigma E\,.$$

Thus

$$\boxed{\omega \ll 4\pi\sigma\,.} \tag{6.31}$$

In the same order to the small parameter ω/σ, we neglect the **convective** current in comparison with the **conductive** current in Ohm's law.

Actually,

$$\rho^{\,\mathrm{q}}\, v \approx v \operatorname{div}\mathbf{E}\,\frac{1}{4\pi} \approx \frac{L}{\tau}\,\frac{E}{L}\,\frac{1}{4\pi} \approx \frac{\omega}{4\pi}\,E \ll \sigma E\,,$$

once the condition (6.31) is satisfied.

The conductivity of astrophysical plasma is often very high (Exercise 5.1).

This is why condition (6.31) is satisfied up to frequencies close to optical ones.

Neglecting the displacement current and the convective current, Maxwell's equations and Ohm's law result in the following relations:

$$\mathbf{j} = \frac{c}{4\pi}\,\mathrm{rot}\,\mathbf{B}\,, \tag{6.32}$$

$$\mathbf{E} = -\,\frac{1}{c}\;\mathbf{v}\times\mathbf{B} + \frac{c}{4\pi\sigma}\;\mathrm{rot}\,\mathbf{B}\,, \tag{6.33}$$

$$\rho^{\,\mathrm{q}} = -\,\frac{1}{4\pi c}\;\mathrm{div}\,(\,\mathbf{v}\times\mathbf{B}\,)\,, \tag{6.34}$$

$$\mathrm{div}\,\mathbf{B} = 0\,, \tag{6.35}$$

$$\frac{\partial\mathbf{B}}{\partial t} = \mathrm{rot}\,(\,\mathbf{v}\times\mathbf{B}\,)\, + \frac{c^2}{4\pi\sigma}\;\Delta\,\mathbf{B}\,. \tag{6.36}$$

Once $\mathbf{B}$ and $\mathbf{v}$ are given, the current $\mathbf{j}$, the electric field $\mathbf{E}$, and the volume charge $\rho^{\,\mathrm{q}}$ are determined by formulae (6.32)—(6.34).

Thus

> the problem is reduced to finding the interaction of two vector fields: the magnetic field $\mathbf{B}$ and the hydrodynamic velocity field $\mathbf{v}$.

As a consequence, the approach under discussion is known as **magnetohydrodynamics** (MHD).

The corresponding equation of motion is obtained by substitution of (6.32)–(6.34) in the equation of momentum transfer (6.5).

With the viscous forces as usually written in hydrodynamics, we have

$$\rho \frac{d\mathbf{v}}{dt} = -\nabla p + \rho^{\mathrm{q}} \mathbf{E} - \frac{1}{4\pi} \mathbf{B} \times \operatorname{rot} \mathbf{B} +$$

$$+ \eta \, \Delta \, \mathbf{v} + \left(\zeta + \frac{\eta}{3} \right) \nabla \operatorname{div} \mathbf{v} \, . \tag{6.37}$$

Here η is the **first viscosity** coefficient, ζ is the **second viscosity** coefficient (see Landau and Lifshitz, *Fluid Mechanics*).

Formulae for these coefficients and the viscous forces should be derived from the moment equation for the pressure tensor.

* * *

The second additional assumption has to be introduced now.

Treating Equation (6.37), the **electric force** $\rho^{\mathrm{q}} \mathbf{E}$ can be ignored in comparison to the magnetic one if

$$v^2 \ll c^2 \, , \tag{6.38}$$

that is in the non-relativistic approximation.

To make certain that this is true, evaluate the electric force

$$\rho^{\mathrm{q}} E \approx \frac{1}{4\pi c} \frac{vB}{L} \frac{vB}{c} \approx \frac{B^2}{4\pi} \frac{1}{L} \frac{v^2}{c^2} \tag{6.39}$$

and the magnetic force

$$\frac{1}{4\pi}\,|\,\mathbf{B}\times\mathrm{rot}\,\mathbf{B}\,|\approx\frac{B^2}{4\pi}\,\frac{1}{L}\,. \tag{6.40}$$

Comparing (6.39) with (6.40), we see that the electric force is a factor of v^2/c^2 short of the magnetic one.

In a great number of astrophysical applications, the plasma velocities fall far short of the speed of light.

The Sun is a good case in point.

The largest velocities in **coronal mass ejections** (CMEs) do not exceed 3×10^8 cm/s.

Thus,

we neglect the electric force acting upon the volume charge in comparison with the magnetic force.

However the **relativistic objects** like accretion disks near rotating black holes (Novikov and Frolov, 1989), and **pulsar** magnetospheres are at the other extreme.

The electric force plays a crucial role in electrodynamics of relativistic objects.

6.2.3 Non-relativistic MHD

With the assumptions made above (2 + 3 + 2),

the considerable simplifications have been obtained;

and now we write the following **set of equations** of non-relativistic MHD:

$$\frac{\partial}{\partial t}\,\rho\, v_\alpha = -\frac{\partial}{\partial r_\beta}\,\Pi^*_{\alpha\beta}\,, \tag{6.41}$$

$$\frac{\partial \mathbf{B}}{\partial t} = \text{rot}\,(\,\mathbf{v}\times\mathbf{B}\,) + \nu_{\text{m}}\,\Delta\,\mathbf{B}\,, \tag{6.42}$$

$$\text{div}\,\mathbf{B} = 0\,, \tag{6.43}$$

$$\frac{\partial \rho}{\partial t} + \text{div}\,\rho\,\mathbf{v} = 0\,, \tag{6.44}$$

$$\frac{\partial}{\partial t}\left(\frac{\rho v^2}{2} + \rho\varepsilon + \frac{B^2}{8\pi}\right) = -\,\text{div}\,\mathbf{G}\,, \tag{6.45}$$

$$p = p\,(\rho, T)\,. \tag{6.46}$$

The momentum of electromagnetic field does **not** appear on the left-hand side of (6.41).

It is negligibly small in comparison to the plasma momentum $\rho\, v_\alpha$.

This fact is a consequence of neglecting the displacement current.

On the right-hand side of (6.41), the asterisk refers to the total momentum flux tensor $\Pi^*_{\alpha\beta}$, which equals

$$\Pi^*_{\alpha\beta} = \rho\, v_\alpha v_\beta + \left(p\,\delta_{\alpha\beta} - \sigma^{\text{v}}_{\alpha\beta}\right) +$$

$$+\frac{1}{4\pi}\left(\frac{B^2}{2}\,\delta_{\alpha\beta} - B_\alpha B_\beta\right)\,. \tag{6.47}$$

In Equation (6.42)

$$\nu_{\rm m} = \frac{c^2}{4\pi\sigma} \tag{6.48}$$

is the **magnetic viscosity**.

It plays the same role as the kinematic viscosity $\nu = \eta/\rho$ in the equation of motion.

The vector **G** is defined as the energy flux

$$G_\alpha = \rho\, v_\alpha \left(\frac{v^2}{2} + w \right) + \frac{1}{4\pi} \left[\mathbf{B} \times (\mathbf{v} \times \mathbf{B}) \right]_\alpha -$$

$$- \frac{\nu_{\rm m}}{4\pi} (\mathbf{B} \times {\rm rot}\, \mathbf{B})_\alpha - \sigma^{\rm v}_{\alpha\beta}\, v_\beta - \kappa\, \nabla_\alpha T\,. \tag{6.49}$$

The **Poynting vector** as a part in (6.49) is

$$\mathbf{G}_{\rm P} = \frac{1}{4\pi}\, \mathbf{B} \times (\mathbf{v} \times \mathbf{B}) - \frac{\nu_{\rm m}}{4\pi}\, \mathbf{B} \times {\rm rot}\, \mathbf{B}\,. \tag{6.50}$$

The energy flux due to **friction** is written as the contraction of the velocity vector **v** and the viscous stress tensor $\sigma^{\rm v}_{\alpha\beta}$.

6.2.4 Energy conservation

The non-relativistic MHD equations are frequently used to model **solar flares**, eruptive prominences, etc.

A goal of such studies is to deduce **how energy** of magnetic field **is stored** and then suddenly **released** to drive these phenomena.

However

most models use a **simple** energy equation,

the discussion often centers on the **over-simplified** interpretation or

just comparison of magnetic field **structure** in the models with corresponding features observed in emission.

With new capabilities to study X-ray and EUV emission from **Hinode** and complementary observations from **SOHO**, **RHESSI** and other satellites, the models advance to **more quantitative** results.

We have to develop the MHD models that include **radiative losses** and other dissipative processes, the energy transport by **anisotropic heat conduction**.

The equation of state (6.46) can be rewritten in other thermodynamic variables.

In order to do this, we have to make use of the thermodynamic identities

$$d\varepsilon = T\, ds + \frac{p}{\rho^2}\, d\rho \quad \text{and} \quad dw = T\, ds + \frac{1}{\rho}\, dp\,.$$

Here s is the **entropy per unit mass**.

We transform the energy conservation law (6.45) from the divergent form to the hydrodynamic one:

$$\rho\, T\, \frac{ds}{dt} = \frac{\nu_{\mathrm{m}}}{4\pi}\, (\mathrm{rot}\, \mathbf{B})^2 \,+\, \sigma^{\mathrm{v}}_{\alpha\beta}\, \frac{\partial v_\alpha}{\partial r_\beta} + $$
$$+ \,\mathrm{div}\, \kappa \nabla T \,+ \mathcal{L}^{(rad)}\, (\mathbf{r}, t)\,. \qquad (6.51)$$

Thus

> the heat abundance change $dQ = \rho T\, ds$ in a moving element of unit volume is a sum of the Joule and viscous heating, conductive heat redistribution and radiative cooling.

6.2.5 Relativistic magnetohydrodynamics

Relativistic MHD models are of considerable interest in several areas of modern astrophysics.

The theory of gravitational collapse and models of **supernova** explosions are based on **relativistic hydrodynamics** for a star.

The effects of deviations from spherical symmetry due to magnetic field require the use of **relativistic MHD** models.

Relativistic hydrodynamics is presumably applied to the so-called **quark-gluon plasma** which is the primordial state of hadronic matter in the Universe.

When the medium interacts electromagnetically and is **highly conducting**, the simplest description is in terms of relativistic MHD.

From the mathematical viewpoint, the relativistic MHD was mainly treated in the framework of **general relativity**.

This means that the MHD equations were studied in conjunction with Einstein's equations.

Lichnerowicz (1967) has made a thorough and deep investigation of the initial value problem.

In many applications, however, one neglects the gravitational field generated by the conducting medium in comparison with the background gravitational field

as well as

in many cases one simply uses **special relativity**.

Such relativistic MHD is much simpler than the full general relativistic theory.

So more detailed results can be obtained (Novikov and Frolov, 1989).

6.3 Magnetic flux conservation. Ideal MHD

6.3.1 Integral and differential forms of the law

Equations (6.44), (6.41), and (6.45) are the conservation laws for mass, momentum, and energy, respectively.

Let us show that Equation (6.42):

$$\frac{\partial \mathbf{B}}{\partial t} = \text{rot}\,(\,\mathbf{v} \times \mathbf{B}\,) + \nu_{\text{m}}\,\Delta\,\mathbf{B}\,,$$

with $\nu_{\text{m}} = 0$, is the **magnetic flux conservation** law.

Let us consider the time derivative of the vector **B** flux through a surface S moving with the plasma (Fig. 6.2).

According to the known formula of vector analysis (see Smirnov, 1965), we have

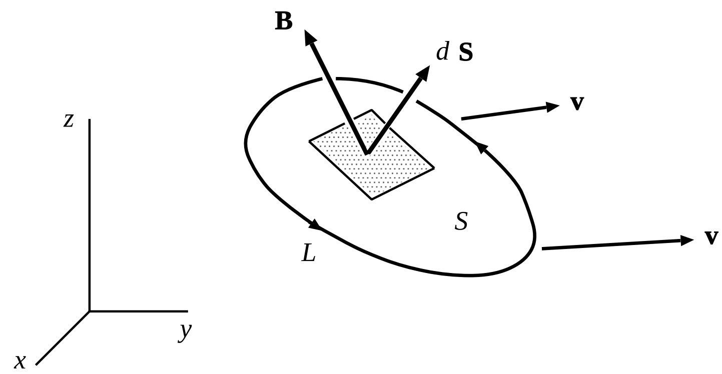


Figure 6.2: The magnetic field $\mathbf{B}$ flux through the surface S moving with a plasma with velocity $\mathbf{v}$.

$$\frac{d}{dt}\int_S \mathbf{B}\cdot d\,\mathbf{S} = \int_S \left(\frac{\partial\,\mathbf{B}}{\partial t} + \mathbf{v}\,\mathrm{div}\,\mathbf{B} + \mathrm{rot}\,(\,\mathbf{B}\times\mathbf{v}\,)\right)\cdot d\,\mathbf{S}\,.$$

Since $\mathrm{div}\,\mathbf{B} = 0$,

$$\frac{d}{dt}\int_S \,\mathbf{B}\cdot d\,\mathbf{S} = \int_S \left(\frac{\partial\,\mathbf{B}}{\partial t} - \mathrm{rot}\,(\,\mathbf{v}\times\mathbf{B}\,)\right)\cdot d\,\mathbf{S}\,,$$

or, making use of Equation (6.42),

$$\boxed{\frac{d}{dt}\int_S \,\mathbf{B}\cdot d\,\mathbf{S} = \nu_{\mathrm{m}}\int_S \Delta\mathbf{B}\cdot d\,\mathbf{S}\,.} \qquad (6.52)$$

Thus, if we cannot neglect magnetic viscosity ν_{m}, then

> **the change rate of magnetic flux** through a surface moving together with a conducting plasma is **proportional to the magnetic viscosity**.

The right-hand side of (6.52) can be rewritten with the help of the Stokes theorem:

$$\frac{d}{dt}\int_S \mathbf{B}\cdot d\,\mathbf{S} = -\,\nu_{\mathrm{m}}\oint_L \mathrm{rot}\,\mathbf{B}\cdot d\,\mathbf{l}\,.$$

Here L is the '**liquid contour**' bounding the surface S.

By using equation

$$\mathbf{j} = \frac{c}{4\pi}\,\mathrm{rot}\,\mathbf{B}\,,$$

we have

$$\boxed{\frac{d}{dt}\int_S \mathbf{B}\cdot d\,\mathbf{S} = -\,\frac{c}{\sigma}\oint_L \mathbf{j}\cdot d\,\mathbf{l}\,.} \tag{6.53}$$

The change rate of flux is proportional to **resistivity** σ^{-1} of the plasma.

Equation (6.53) is equivalent to the differential Equation (6.42) and presents an **integral** form of the magnetic flux conservation law.

> **The magnetic flux** through any surface moving with the plasma **is conserved**, once the electric resistivity σ^{-1} can be ignored.

When is it possible to neglect resistivity of plasma?

The relative role of a dissipation process can be evaluated as follows.

Let us pass on to the dimensionless variables

$$\mathbf{r}^* = \frac{\mathbf{r}}{L}\,, \qquad t^* = \frac{t}{\tau}\,, \qquad \mathbf{v}^* = \frac{\mathbf{v}}{v}\,, \qquad \mathbf{B}^* = \frac{\mathbf{B}}{B_0}\,.$$

On substituting them in (6.42) we obtain

$$\frac{B_0}{\tau}\,\frac{\partial\,\mathbf{B}^*}{\partial t^*} = \frac{vB_0}{L}\;\mathrm{rot}^*\,(\,\mathbf{v}^* \times \mathbf{B}^*\,) + \nu_{\mathrm{m}}\,\frac{B_0}{L^2}\;\Delta^*\,\mathbf{B}^*\,.$$

Now we normalize this equation with respect to its left-hand side, i.e.

$$\frac{\partial\,\mathbf{B}^*}{\partial t^*} = \frac{v\tau}{L}\;\mathrm{rot}^*\,(\,\mathbf{v}^* \times \mathbf{B}^*\,) + \frac{\nu_{\mathrm{m}}\tau}{L^2}\,\Delta^*\,\mathbf{B}^*\,.$$

This dimensionless equation contains **two dimensionless parameters**.

The first one,

$$\delta = \frac{v\tau}{L}\,,$$

will be discussed later on.

Here, for simplicity, we assume $\delta = 1$.

The second parameter,

$$\boxed{\mathrm{Re}_{\,\mathrm{m}} = \frac{L^2}{\nu_{\mathrm{m}}\,\tau} = \frac{vL}{\nu_{\mathrm{m}}}\,,} \tag{6.54}$$

is termed the **magnetic** Reynolds number, by analogy with the **hydrodynamic** Reynolds number

$$\mathrm{Re} = \frac{vL}{\nu}\,.$$

Omitting the asterisk, we write the **dimensionless** equation

$$\frac{\partial\,\mathbf{B}}{\partial t} = \mathrm{rot}\,(\,\mathbf{v}\times\mathbf{B}\,) + \frac{1}{\mathrm{Re}_{\mathrm{m}}}\,\Delta\mathbf{B}\,. \tag{6.55}$$

> The larger the magnetic Reynolds number, the smaller the role played by magnetic viscosity.

So the magnetic Reynolds number is the measure of a **relative importance of resistivity**.

If

$$\mathrm{Re}_{\mathrm{m}} \gg 1\,,$$

we neglect the plasma resistivity and, as a consequence, magnetic field **diffusion and dissipation**.

On the contrary, in laboratory, e.g., in devices for studying reconnection, because of a small value L^2, the magnetic Reynolds number is usually not large:

$$\mathrm{Re}_{\mathrm{m}} \sim 1 - 3\,.$$

In this case, the resistivity has a dominant role, and **dissipation** is important.

6.3.2 The ideal MHD

Under astrophysical conditions, owing to the **low resistivity** of plasma and the enormously **large length** scales, the magnetic Reynolds number is usually huge:

$$\mathrm{Re}_{\mathrm{m}} > 10^{10}$$

(e.g., Exercise 6.1).

Thus, in a great number of problems, it is sufficient to consider a medium with **infinite conductivity**:

$$\mathrm{Re}_{\mathrm{m}} \gg 1\,.$$

Furthermore the usual Reynolds number can be also large (see, however, Exercise 6.2):

$$\mathrm{Re} \gg 1\,.$$

Let us also assume the **heat exchange** to be of minor importance.

This assumption is **not** universally true either.

Sometimes the thermal conductivity is so effective that an astrophysical plasma must be considered as **isothermal**, rather than adiabatic.

However, conventionally,

> while treating the '**ideal medium**', all dissipative coefficients as well as the thermal conductivity are set equal to zero:

$$\boxed{\nu_{\rm m} = 0\,, \quad \eta = \zeta = 0\,, \quad \kappa = 0\,.}$$

The complete set of the ideal MHD equations has two equivalent forms.

The first one is the **transfer** equations:

$$\begin{aligned}
&\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\,\mathbf{v} = -\frac{\nabla p}{\rho} - \frac{1}{4\pi\rho}\,\mathbf{B}\times\mathrm{rot}\,\mathbf{B}\,,\\
&\frac{\partial\,\mathbf{B}}{\partial t} = \mathrm{rot}\,(\mathbf{v}\times\mathbf{B})\,, \qquad \mathrm{div}\,\mathbf{B} = 0\,,\\
&\frac{\partial \rho}{\partial t} + \mathrm{div}\,\rho\,\mathbf{v} = 0\,, \qquad \frac{\partial s}{\partial t} + (\mathbf{v}\cdot\nabla)\,s = 0\,,\\
&p = p\,(\rho, s)\,.
\end{aligned} \tag{6.56}$$

The **divergent** form corresponds to the **conservation laws** for energy, momentum, mass and magnetic flux:

$$\frac{\partial}{\partial t}\left(\frac{\rho v^2}{2} + \rho\varepsilon + \frac{B^2}{8\pi}\right) = -\,\mathrm{div}\,\mathbf{G}\,, \tag{6.57}$$

$$\frac{\partial}{\partial t}\,\rho\,v_\alpha = -\,\frac{\partial}{\partial r_\beta}\,\Pi^{*}_{\alpha\beta}\,, \tag{6.58}$$

$$\frac{\partial \rho}{\partial t} = -\,\mathrm{div}\,\rho\,\mathbf{v}\,, \tag{6.59}$$

$$\frac{\partial \mathbf{B}}{\partial t} = \text{rot}\,(\,\mathbf{v} \times \mathbf{B}\,)\,, \tag{6.60}$$

$$\text{div}\,\mathbf{B} = 0\,, \tag{6.61}$$

$$p = p\,(\rho, s)\,. \tag{6.62}$$

Here the energy flux and the momentum flux tensor are

$$\mathbf{G} = \rho\,\mathbf{v}\left(\frac{v^2}{2} + w\right) + \frac{1}{4\pi}\left(B^2\,\mathbf{v} - (\mathbf{B}\cdot\mathbf{v})\,\mathbf{B}\,\right) \tag{6.63}$$

and

$$\Pi^*_{\alpha\beta} = p\,\delta_{\alpha\beta} + \rho\,v_\alpha v_\beta + \frac{1}{4\pi}\left(\frac{B^2}{2}\,\delta_{\alpha\beta} - B_\alpha B_\beta\right). \tag{6.64}$$

6.3.3 The 'frozen field' theorem

The magnetic flux conservation law (6.60) written in the integral form

$$\frac{d}{dt}\int\limits_S \mathbf{B}\cdot d\,\mathbf{S} = 0$$

allows us to represent the magnetic field as a set of **field lines attached to the medium**, as if they were '**frozen into**' it.

For this reason, (6.60) is referred to as the '**freezing-in**' equation.

The "frozen field" theorem can be formulated as follows.

> In the ideally conducting medium, the field lines move together with the plasma. A **medium motion conserves** not only the magnetic flux but **each of the field lines** as well.

Let us imagine a **thin tube** of field lines (Fig. 6.3).

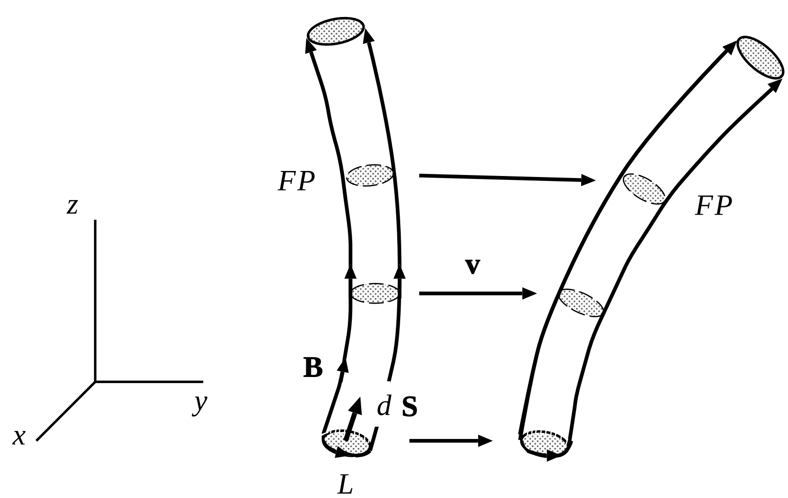


Figure 6.3: The field-flux tube through the surface $d\,\mathbf{S}$ moves with a plasma with velocity $\mathbf{v}$. L is the "**liquid contour**" bounding the surface $d\,\mathbf{S}$. The "**fluid particles**" (FP) that are initially in this flux tube remain in the same tube.

There is **no** magnetic flux through any part of the surface formed by the boundary field lines that intersect the closed curve L.

Hence, **the "fluid particles"** that are initially in the same flux tube must **remain in the flux tube**.

In ideal MHD flows, magnetic field lines are therefore **materialized** and are **unbreakable** because the flux tube links the same fluid particles.

As a result its **topology cannot change**.

Fluid particles which are **not** initially on a common field line **cannot** become linked by one later on.

> This general topological constraint **restricts the ideal MHD motions**, forbidding a lot of motions that would otherwise appear.

Conversely, the fluid particle motion, whatever its complexity, may create situations where the magnetic field structure becomes itself **very complex**.

* * *

In general, the field intensity **B** is a **local** quantity.

However the magnetic field lines (even in vacuum) are **integral** characteristics of the field.

Their analysis becomes more complicated.

Nonetheless, an investigation of **non-local** structures of magnetic fields is fairly important in plasma astrophysics.

The **geometry** of the field lines appears in different ways in the equilibrium criteria for astrophysical plasma.

Much depends on whether the field lines are **concave** or **convex**, on the so-called **specific volume** of magnetic flux tubes.

However even more depends on the presence of X-type points, as well as on other **topological** characteristics, e.g. the global **magnetic helicity**.

6.4 Magnetic reconnection

Reconnection of magnetic field lines is the physical process which involves a breakdown of the "**frozen field**" theorem.

The effects of electric resistivity, normally negligible in the large, **become locally dominant**

with dramatic consequences in the large-scale plasma flows and magnetic field configuration.

Reconnection changes topology of magnetic field.

The origin of the concept of reconnection lies in an attempt by Giovanelli (1946) to explain **solar flares**.

Reconnection is the means by which **energy stored** in magnetic fields **is released** rapidly to produce such phenomena as solar flares and magnetospheric substorms.

Furthermore, reconnection plays important roles in many areas of astrophysics.

Depending on **complexity** of fields and conditions, reconnection can occur over an **extended region** in space or can be "**patchy**" and "**unpredictable**".

For example, in the Earth's magnetosphere the **reconnecting current layers** (RCLs) are formed by the interaction between the solar wind and the geomagnetic field.

Such RCLs have finite extents, and their **boundary conditions** often change rapidly.

On the contrary,

> **in the solar wind**, the magnetic field orientations on the two sides of the **interplanetary current layers** are usually well defined, and the boundary conditions seem to be relatively stable.

Phan et al. (2006) report the 3-spacecraft observations of plasma flow associated with **large-scale reconnection** in the solar wind (Fig. 6.4).

> In the most astrophysical situations, the reconnection process is predictable and occurs in an **internal scale** of a phenomenon, which is responsible to the initial and boundary conditions.

In the solar wind the scale of a current layer (CL) around the Sun can be very large (Fig. 6.5).

The current layer (CL) separates the fields of nearly opposite directions.

The average plane of the layer is approximately the plane of the equator of the Sun's average magnetic dipole (M) field.

On the other hand, the high-speed solar wind that originates in **coronal holes** is permeated by evolved Alfvén-type fluctuations associated with MHD turbulence.

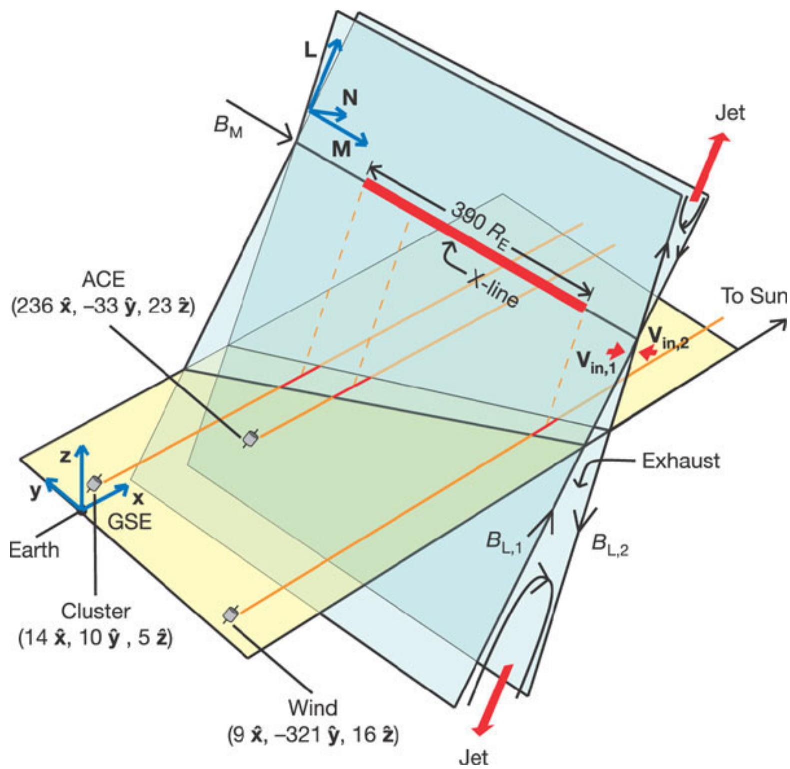


Figure 6.4: The *Wind*, *ACE* and *Cluster* spacecraft on 2 February 2002: The spacecraft positions are shown in units of Earth radius (R_E) and in geocentric solar ecliptic coordinates.

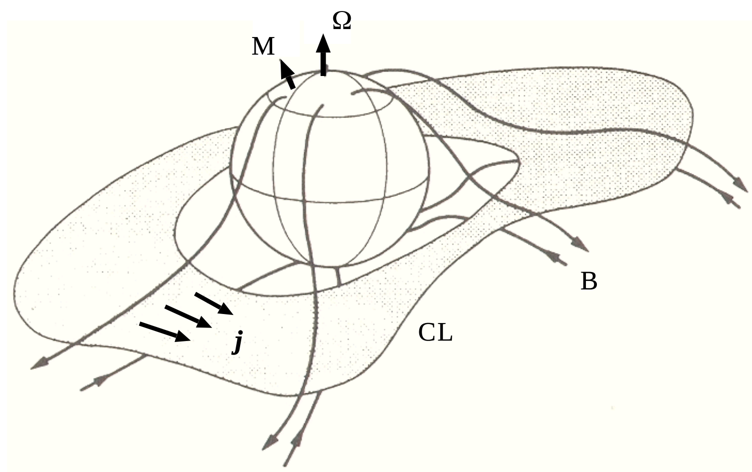


Figure 6.5: A "**wavy ecliptic current**" layer (CL). The Sun is the center of an extensive layer.

The spacecraft *Wind* allows us to study reconnection in this turbulent flow.

The *Wind* observations demonstrate that

> reconnection is one way in which the solar wind **turbulence is dissipated** and the high-speed wind is heated far from the Sun.

In the solar wind, the kinetic and thermal energies of plasma exceed the magnetic energy.

We neglect the magnetic force as compared to the inertia force of moving plasma and its pressure gradient.

We call such a process as reconnection in a **weak magnetic field**.

Another example of this phenomenon is the photospheric reconnection.

Reconnection in a **strong magnetic field** is a fundamental feature of astrophysical plasmas like the solar corona.

Such reconnection explains an accumulation of magnetic energy and a sudden release of this energy, a **flare**.

This phenomenon is accompanied by **fast ejections** of plasma, powerful flows of heat and hard electromagnetic radiation, by **acceleration of particles**.

6.5 Practice: Exercises and Answers

Exercise 6.1. Estimate the magnetic Reynolds number in the **solar corona**.

Answer.

Taking characteristic values of the parallel conductivity as estimated in Exercise 5.1:

$$\sigma_{\parallel} = \sigma \sim 10^{16} - 10^{17}\ \mathrm{s}^{-1}\,,$$

we obtain

$$\mathrm{Re}_{\mathrm{m}} = \frac{vL}{\nu_{\mathrm{m}}} \sim 10^{11} - 10^{12}\,, \tag{6.65}$$

if the length and velocity, $L \sim 10^4$ km and $v \sim 10$ km s^{-1}.

Exercise 6.2. Show that

in the solar corona, **usual viscosity** of plasma can be a much more important dissipative mechanism than electric resistivity.

Answer.

The characteristic value of **kinematic viscosity**

$$\nu = \frac{\eta}{\rho} \approx 3 \times 10^{15} \text{ cm}^2 \, \text{s}^{-1}.$$

Here $T_{\rm p} \approx 2 \times 10^6$ K and $n_{\rm p} \approx n_{\rm e} \approx 2 \times 10^8 \, \text{cm}^{-3}$ were taken as the typical **proton temperature** and density.

If the length and velocity, $L \sim 10^9$ cm and $v \sim 10^6$ cm s^{-1}, then the ordinary Reynolds number

$$\text{Re} = \frac{vL}{\nu} \sim 0.3 \, . \tag{6.66}$$

Thus

$$\text{Re}_{\rm m} \gg \text{Re} \, .$$

Chapter 7

MHD in Astrophysics

> MHD is appropriate for many phenomena in astrophysical plasma, that take place on a relatively large scale.
>
> The non-relativistic MHD is applied to dynamo theory, flows in the solar atmosphere, flares, coronal heating, solar and stellar winds.
>
> Relativistic MHD describes well accretion disks near relativistic objects, and relativistic jets.

7.1 The main approximations in ideal MHD

7.1.1 Dimensionless equations

The equations of MHD constitute a set of **nonlinear differential** equations in **partial derivatives**.

The order of the set is rather high, and its structure is complicated.

To formulate a problem, we have to know the **initial and boundary conditions** admissible by this set of equations.

To do this, in turn, we have to know the **type of equations**, in the sense adopted in mathematical physics.

To formulate a problem, we usually use one or another approximation, which makes it possible to point up and isolate the **main effect**.

For instance, if the **magnetic Reynolds number is small**, a plasma moves comparatively easily with respect to magnetic field.

This is the case in laboratory and technical devices.

The opposite approximation is that of **large magnetic Reynolds numbers**, when the magnetic field 'freezing in' takes place in plasma.

This approximation is quite characteristic of the astrophysical plasma.

How can we isolate the main effect in a phenomenon and correctly formulate the problem? – From the above examples, the following rule suggests itself:

> take the dimensional parameters of a phenomenon, combine them into dimensionless combinations, calculate their values, and use a corresponding approximation in the **dimensionless** equations.

Such an approach is effective in hydrodynamics.

Let us start with the ideal MHD equations:

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\, \mathbf{v} = -\frac{\nabla p}{\rho} - \frac{1}{4\pi\rho}\, \mathbf{B} \times \operatorname{rot} \mathbf{B}\,, \tag{7.1}$$

$$\frac{\partial\, \mathbf{B}}{\partial t} = \operatorname{rot}\, (\mathbf{v} \times \mathbf{B})\,, \tag{7.2}$$

$$\frac{\partial \rho}{\partial t} + \operatorname{div} \rho\, \mathbf{v} = 0\,, \tag{7.3}$$

$$\frac{\partial s}{\partial t} + (\mathbf{v} \cdot \nabla)\, s = 0\,, \tag{7.4}$$

$$\operatorname{div} \mathbf{B} = 0\,, \tag{7.5}$$

$$p = p\, (\rho, s)\,. \tag{7.6}$$

Let the quantities

$$L,\ \tau,\ v,\ \rho_0\,,\ p_0\,,\ s_0\,,\ B_0$$

be the characteristic values of length, time, velocity, density, pressure, entropy and field strength, respectively.

Rewrite Equations (7.1)–(7.6) in the dimensionless variables

$$\mathbf{r}^* = \frac{\mathbf{r}}{L}\,, \quad t^* = \frac{t}{\tau}\,, \ \dots \quad \mathbf{B}^* = \frac{\mathbf{B}}{B_0}\,.$$

Omitting the asterisk, we obtain the equations in dimensionless variables:

$$\varepsilon^2 \left\{ \frac{1}{\delta} \, \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \, \mathbf{v} \right\} = - \, \gamma^2 \, \frac{\nabla p}{\rho} - \frac{1}{\rho} \, \mathbf{B} \times \mathrm{rot} \, \mathbf{B} \, , \quad (7.7)$$

$$\frac{\partial \, \mathbf{B}}{\partial t} = \delta \, \mathrm{rot} \, (\mathbf{v} \times \mathbf{B}) \, , \quad (7.8)$$

$$\frac{\partial \rho}{\partial t} + \delta \, \mathrm{div} \, \rho \, \mathbf{v} = 0 \, , \quad (7.9)$$

$$\frac{\partial s}{\partial t} + \delta \, (\mathbf{v} \cdot \nabla) \, s = 0 \, , \quad (7.10)$$

$$\mathrm{div} \, \mathbf{B} = 0 \, , \quad (7.11)$$

$$p = p \, (\rho, s) \, . \quad (7.12)$$

Here

$$\boxed{\delta = \frac{v \tau}{L} \, , \quad \varepsilon^2 = \frac{v^2}{V_{\mathrm{A}}^{\,2}} \, , \quad \gamma^2 = \frac{p_0}{\rho_0 V_{\mathrm{A}}^{\,2}}} \quad (7.13)$$

are **three dimensionless parameters**;

$$V_{\mathrm{A}} = \frac{B_0}{\sqrt{4 \pi \rho_0}} \quad (7.14)$$

is the characteristic value of the Alfvén speed (see Exercise 7.1).

If the gravitational force is taken into account, Equation (7.7) contains another dimensionless parameter,

$$gL/V_{\mathrm{A}}^{2}\,,$$

where g is the gravitational acceleration.

The analysis of these parameters allows us to separate the approximations which are possible in the ideal MHD.

7.1.2 Weak magnetic fields in astrophysical plasma

We begin with the assumption that

$$\varepsilon^2 \gg 1 \qquad \text{and} \qquad \gamma^2 \gg 1\,. \tag{7.15}$$

As is seen from Equation (7.7), in the zero-order approximation relative to the **small parameters**

$$\varepsilon^{-2} \quad \text{and} \quad \gamma^{-2}\,,$$

we **neglect the magnetic force** as compared to the inertia force and the pressure gradient.

In subsequent approximations, the magnetic effects are treated as **small corrections** to the hydrodynamic ones.

A lot of problems of plasma astrophysics are solved in this approximation, termed the **weak** magnetic field approximation.

Among the simplest of them are the ones concerning the weak field's influence on **hydrostatic equilibrium**.

An example is the problem of the influence of magnetic field on the equilibrium of a self-gravitating plasma ball (a star, the magnetoid of quasar's kernel etc.).

Some other problems are in fact analogous to the mentioned ones.

They are called **kinematic** problems, since

> they treat the influence of a given plasma flow on magnetic field; the reverse influence is considered to be negligible.

Such problems are reduced to finding the magnetic field resulting from the known velocity field.

An example is the magnetic field amplification and support by stationary plasma flows (**magnetic dynamo**).

The simplest example is the magnetic field amplification by **differential rotation**.

A leading candidate to explain the origin of large-scale magnetic fields in astrophysical plasma is the **turbulent dynamo** theory.

7.1.3 Strong magnetic fields in plasma

The opposite approximation – that of the **strong** field – reflects the specificity of MHD to a greater extent.

This approximation is valid when the **magnetic force**

$$\mathbf{F}_m = -\frac{1}{4\pi}\,\mathbf{B} \times \operatorname{rot}\mathbf{B} \tag{7.16}$$

dominates all the others (inertia force, pressure gradient, etc.).

In Equation (7.7), the magnetic field is a **strong** one if

$$\varepsilon^2 \ll 1 \qquad \text{and} \qquad \gamma^2 \ll 1\,, \tag{7.17}$$

i.e. the magnetic energy density greatly exceeds that of the kinetic and thermal energies:

$$\frac{B_0^2}{8\pi} \gg \frac{\rho_0 v^2}{2} \qquad \text{and} \qquad \frac{B_0^2}{8\pi} \gg 2 n_0 k_{\text{B}} T_0\,.$$

From Equation (7.7) it follows that, in the zeroth order with respect to the small parameters (7.17), the magnetic field is **force-free**:

$$\mathbf{B} \times \text{rot}\,\mathbf{B} = 0\,. \tag{7.18}$$

This conclusion is quite natural:

> if the magnetic force dominates all the others, the magnetic field must balance itself in the region under consideration.

Condition (7.18) means that electric currents flow parallel to magnetic field lines.

If, in addition, electric currents are absent in some region, then the strong field is simply **potential**:

$$\text{rot}\,\mathbf{B} = 0\,, \qquad \mathbf{B} = \nabla\Psi\,, \qquad \Delta\Psi = 0\,. \tag{7.19}$$

Let us consider the **first order** in the small parameters (7.17).

If they are not equally significant, there are **two possibilities**.

(a) We suppose, at first, that

$$\varepsilon^2 \ll \gamma^2 \ll 1\,. \tag{7.20}$$

Then we neglect the inertia force in Equation (7.7) as compared to the pressure gradient.

Decomposing the magnetic force into a **magnetic tension** force and a **magnetic pressure** gradient,

$$\mathbf{F}_m = -\frac{1}{4\pi}\,\mathbf{B} \times \mathrm{rot}\,\mathbf{B} \;=\; \frac{1}{4\pi}\,(\mathbf{B}\cdot\nabla)\,\mathbf{B} - \nabla\,\frac{B^2}{8\pi}\,, \tag{7.21}$$

we obtain the following dimensionless equation:

$$(\mathbf{B}\cdot\nabla)\,\mathbf{B} = \nabla\left(\frac{B^2}{2} + \gamma^2 p\right). \tag{7.22}$$

Owing to the **gas pressure** gradient, the magnetic field differs from the force-free one:

> the magnetic tension force $(\mathbf{B}\cdot\nabla)\,\mathbf{B}/4\pi$ must balance not only the magnetic pressure gradient but that of the **gas pressure** as well.

Obviously the effect is proportional to the small parameter γ^2.

This approximation is naturally called the **magnetostatics** since $\mathbf{v} = 0$.

It works in regions of a strong magnetic field where the gas pressure gradients are large, e.g., in **coronal loops** and **reconnecting current layers** in the solar corona.

(b) The inertia force also causes the magnetic field to deviate from the force-free one:

$$\varepsilon^2 \left\{ \frac{1}{\delta} \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\, \mathbf{v} \right\} = - \frac{1}{\rho}\, \mathbf{B} \times \operatorname{rot} \mathbf{B} \, . \tag{7.23}$$

Here we ignored the pressure gradient as compared with the inertia force.

This is the case

$$\gamma^2 \ll \varepsilon^2 \ll 1 \, . \tag{7.24}$$

The approximation corresponding (7.24) is termed the approximation of **strong** field and **cold** plasma.

The main applications of this approximation are the **solar atmosphere** and the Earth's **magnetosphere**.

Both objects are **well studied** from the observational viewpoint.

So we can proceed with confidence from **qualitative interpretation** to the construction of **quantitative models**.

The presence of a strong field and a rarefied plasma is common for both phenomena.

> A sufficiently strong magnetic field **easily moves** a comparatively rarefied plasma in many non-stationary phenomena in space.

Some astrophysical applications will be discussed in the following two Sections.

* * *

In closing, let us consider the dimensionless parameter

$$\delta = v\tau / L \,.$$

It characterizes the relative role of
the **local** $\partial/\partial t$
and **transport** $(\mathbf{v} \cdot \nabla)$
terms in the substantial derivative

$$\frac{d}{dt} = \frac{1}{\delta}\,\frac{\partial}{\partial t} + (\mathbf{v} \cdot \nabla)\,.$$

If $\delta \gg 1$, the flow can be considered to be **stationary**

$$\varepsilon^2\,(\mathbf{v} \cdot \nabla)\,\mathbf{v} = -\,\frac{1}{\rho}\;\mathbf{B} \times \mathrm{rot}\,\mathbf{B}\,. \tag{7.25}$$

If $\delta \ll 1$, the transport term $(\mathbf{v} \cdot \nabla)$ can be ignored, and the equation of motion takes the form

$$\varepsilon^2\,\frac{\partial \mathbf{v}}{\partial t} = -\,\frac{1}{\rho}\;\mathbf{B} \times \mathrm{rot}\,\mathbf{B}\,, \tag{7.26}$$

other equations becoming linear.

This case corresponds to **small small perturbations**.

If need be, the right-hand side of Equation (7.26) can be linearized too.

Generally the parameter

$$\delta \approx 1\,,$$

and the MHD equations in the approximation of **strong** field and **cold** plasma take the following dimensionless form:

$$\varepsilon^2\,\frac{d\mathbf{v}}{dt} = -\,\frac{1}{\rho}\,\mathbf{B}\times\mathrm{rot}\,\mathbf{B}\,, \qquad (7.27)$$

$$\frac{\partial\,\mathbf{B}}{\partial t} = \mathrm{rot}\,(\,\mathbf{v}\times\mathbf{B}\,)\,, \qquad (7.28)$$

$$\frac{\partial\rho}{\partial t} + \mathrm{div}\,\rho\,\mathbf{v} = 0\,. \qquad (7.29)$$

In the next Chapter we shall consider some continuous flows, which are described by these equations.

7.2 Accretion disks of stars

7.2.1 Angular momentum transfer

Magnetic fields are discussed as a means of angular transport in the **accretion disk**.

Interest in the magnetic fields in **binary** stars steadily increased after the discovery of the nature of AM Herculis.

The **optical** counterpart of the soft X-ray source has **polarization** in the V and I spectral bands.

This suggested the presence of a strong field, $B \sim 10^8\,\mathrm{G}$, assuming the fundamental cyclotron frequency to be observed.

Other similar systems were soon discovered.

Evidence for strong fields was found in the **X-ray binary pulsars** and the **intermediate polar binaries**.

MHD in binary stars is now an area of central importance in stellar astrophysics (Campbell, 1997).

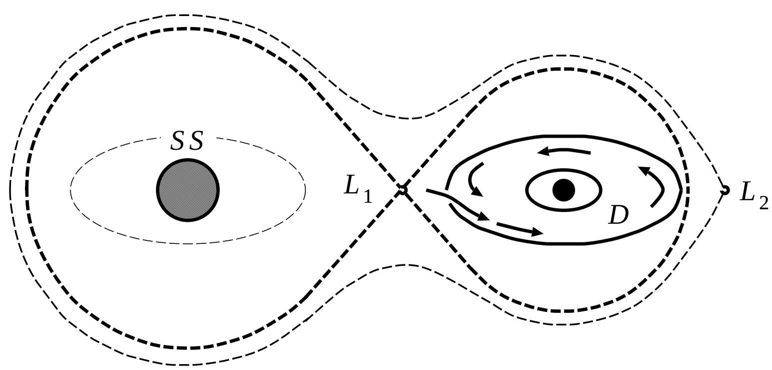


Figure 7.1: A binary system with an accretion disk. The tidally and rotationally distorted secondary star SS loses plasma from the unstable L_1 point. The resulting plasma stream feeds an accretion disk D, centered on the primary star.

The disk is fed by the plasma stream originated in the L_1 region (Fig. 7.1) of the secondary star.

In a steady state,

> plasma is transported through the disk at the rate it is supplied by the stream and the angular momentum is advected outwards.

Such advection requires coupling between rings of rotating plasma; the **ordinary viscosity is too weak** to provide this.

Hence some form of **anomalous viscosity** must be invoked.

Purely **hydrodynamic turbulence** does **not** produce sustained outward transport of angular momentum.

MHD turbulence greatly enhances angular momentum transport (Balbus and Papaloizou, 1999).

> **Turbulent viscous** and **magnetic stresses** cause radial advection of the angular momentum via the **azimuthal forces**.

7.2.2 Accretion in cataclysmic variables

Cataclysmic variables (CVs) are binary systems composed of a **white dwarf** (primary star) and a **late-type**, main-sequence companion (secondary star).

The way this plasma falls towards the primary depends on the **intensity of a magnetic field** of the white dwarf.

The **strong** field ($B \gtrsim 10^7$ G) may entirely dominate the accretion flow.

The magnetic field is strong enough to **synchronize** the white dwarf rotation (spin) with the orbital period.

No disk is formed.

Instead, the field channels accretion towards its **polar regions**.

Such synchronous systems are known as AM Herculis binaries or **polars**.

The **intermediate** ($B \sim 2 - 8 \times 10^6$ G) field primary stars harbor magnetically **truncated** accretion disks which extend until magnetic pressure begins to dominate.

Presumably the plasma is finally accreted onto the **magnetic poles** of the white dwarf.

The asynchronous systems are known as DQ Herculis binaries or **intermediate polars** (IPs).

The accretion geometry strongly influences the **emission** properties at all wavelengths and its variability.

The knowledge of the behavior in all energy domains can allow one to locate the different accreting regions.

The white dwarf LHS 2534 offers the first empirical data of the **Zeeman effect** on neutral Na, Mg, and both ionized and neutral Ca.

The Na I splitting results in a field strength estimate of 1.92×10^6 G.

7.2.3 Accretion disks near black holes

In the binary stars discussed above, there is an **abundance of evidence** for accretion disks:

(a) double-peaked emission lines,

(b) eclipses of an extended light source centered on the primary,

(c) eclipses of the secondary star by the disk.

The case of accretion disks in **active galactic nuclei** (AGN) is less clear.

Nonetheless the disk accretion onto a **super-massive black hole** is the commonly accepted model for these objects.

As the plasma accretes in the gravitational field of the central mass, magnetic field lines are convected inwards, amplified and finally deposited on **the horizon of the black hole**.

As long as a magnetic field is confined by the disk, a **differential rotation** causes the field to wrap up tightly, becoming highly sheared and predominantly azimuthal in orientation.

A dynamo in the disk may be responsible for the maintenance and amplification of the magnetic field.

In the **standard model** of an accretion disk (Shakura and Sunyaev, 1973; Novikov and Thorne, 1973), the gravitational energy is locally radiated from the **optically thin** disk.

However **the expected power far exceeds the observed luminosity**.

There are **two possible explanations**:

(a) the accretion occurs at extremely low rates, or

(b) the accretion occurs at **low radiative efficiency**.

Advection results in a structure different from the standard model.

The advection process physically means that

> the energy generated via viscous dissipation is restored as **entropy** of the accreting **plasma flow** rather than being radiated.

An optically thin **advection-dominated accretion flow** (ADAF) seemed to be a model that can reproduce the observed spectra of black hole systems such as AGN and Galactic black hole candidates.

7.2.4 Flares in accretion disk coronae

Following the launch of several X-ray satellites, astrophysicists have tried to observe and analyze the variations of high energy flux from black hole candidates.

It has appeared that

> there are many **relationships** between flares in the solar corona and 'X-ray shots' in accretion disks.

For example, the peak interval distribution of Cyg X-1 shows that the **occurrence frequency** of large X-ray shots is reduced.

A second large shot does **not** occur soon after a previous large shot.

This suggests the existence of **energy-accumulation structures**, such as non-potential magnetic fields in the solar corona.

It is likely that **accretion disks have a corona**.

Galeev et al. (1979) suggested that the corona is confined in strong magnetic loops which have **buoyantly emerged** from the disk.

> **Magnetic reconnection** of buoyant fields in the lower density surface regions may supply the energy source for a hot corona.

The existence of a disk corona with a **strong** field raises the possibility of a **wind flow** similar to the solar wind.

This would result in **angular momentum transport** away from the disk, which could have some influence on the inflow.

Another feature is the possibility of a **flare energy release** similar to solar flares.

When a plasma in the disk corona is **optically thin** and has a **dominant magnetic pressure**, the circumstances are similar to the solar corona.

Therefore

> it is possible to imagine some **similarity** between the mechanisms of solar flares and X-ray shots in accretion disk coronae.

Besides the effect of heating the the disk corona, reconnection is able to **accelerate particles** to high energies.

Some geometrical and physical properties of the flares in disk coronae can be inferred from X-ray observations of Galactic black hole candidates.

7.3 Astrophysical jets

7.3.1 Jets near black holes

Jet-like phenomena, including relativistic jets, are observed on a wide range of scales in accretion disk systems.

AGN show **extremely energetic outflows** extending beyond the outer edge of a galaxy in the form of **strongly collimated** jets.

There is evidence that **magnetic forces are involved in the driving mechanism** and that the magnetic fields also provide the collimation of **relativistic** flows.

Rotating black holes are thought to be the prime-mover in centers of galaxies.

The gravitational field of rotating black holes is **more complex** than that of non-rotating ones.

The weak-gravity (far from the hole) low-velocity **coordinate acceleration** of uncharged particle

$$\frac{d^2\mathbf{r}}{dt^2} = \mathbf{g} + \frac{d\mathbf{r}}{dt} \times \mathbf{H}_{gr}\,. \tag{7.30}$$

This looks like the Lorentz force with the electric field $\mathbf{E}$ replaced by $\mathbf{g}$, the magnetic field $\mathbf{B}$ replaced by the vector

$$\mathbf{H}_{gr} = \operatorname{rot} \mathbf{A}_{gr}\,,$$

and the electric charge e replaced by the particle mass m.

These analogies lie behind the words "**gravitoelectric**" and "**gravitomagnetic**" to describe the gravitational acceleration field $\mathbf{g}$ and to describe the "**shift function**" $\mathbf{A}_{gr}$ (Exercise 7.6).

Thus, far from the horizon, the gravitational acceleration

$$\mathbf{g} = -\frac{M}{r^2}\,\mathbf{e}_r \tag{7.31}$$

is the radial Newtonian acceleration, and
the **gravitomagnetic** field

$$\mathbf{H}_{gr} = 2\,\frac{\mathbf{J} - 3\,(\,\mathbf{J}\cdot\mathbf{e}_r)\,\mathbf{e}_r}{r^3} \tag{7.32}$$

is a dipole field.

The role of dipole moment is played by the hole's angular momentum

$$\mathbf{J} = \int (\,\mathbf{r} \times \rho_m \mathbf{v})\; dV\,.$$

> The gravitomagnetic force drives an accretion disk into the hole's equatorial plane and holds it there

(Fig. 7.2).

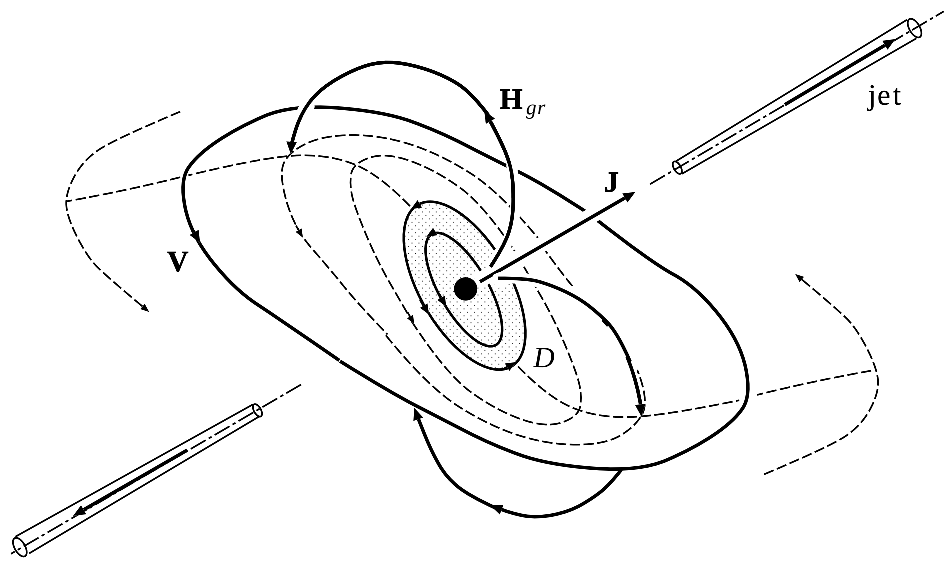


Figure 7.2: An accretion disk D around a rotating black hole is driven into the hole's equatorial plane at small radii by a combination of **gravitomagnetic forces** (action of the gravitomagnetic field $\mathbf{H}_{gr}$ on orbiting plasma) and **viscous** forces.

At radii where the bulk of the disk's gravitational energy is released and where the hole-disk interactions are strong,

> there is **only one geometrically preferred direction** along which a jet might emerge, which coincides with the rotation axis of the black hole.

The jet might be produced by winds off the disk, in other cases by electrodynamic acceleration of the disk, and in others by currents in the hole's magnetosphere.

However whatever the mechanism, the jet presumably is locked to the hole's **rotation axis**.

The black hole acts as a gyroscope to keep the jet aligned.

It is very **difficult to torque a black hole**.

The fact accounts for the constancy of the observed jet directions over length scales as great as **millions of light years** and thus over time scales of millions of years or longer.

* * *

In the **highly-conducting medium**, the gravitomagnetic force couples with electromagnetic fields over Maxwell's equations.

This effect has interesting consequences for the magnetic fields advected towards the black hole.

It leads to a **gravitomagnetic dynamo** which amplifies any seed field near a rotating compact object.

This process builds up the **dipolar magnetic structures** which may be behind the **bipolar outflows** seen as relativistic jets.

7.3.2 Relativistic jets from disk coronae

Relativistic jets are produced perpendicular to the accretion disk plane (Fig. 7.2) around a super-massive black hole in an AGN.

The **shock** of the jets on intergalactic media is considered as being able to accelerate particles up to the highest energies, say 10^{20} eV for **cosmic rays**.

This hypothesis need, however, to be completed by some **necessary ingradients** since such powerful galaxies are rare objects.

The relativistic jets may be powered by **acceleration of protons** in a corona above an accretion disk.

The acceleration arises as a consequence of the shearing motion of the magnetic field lines in the corona, that are anchored in the underlying Keplerian disk.

Particle acceleration in the corona leads to the development of a **pressure-driven wind**.

It passes through a critical point and subsequently transforms into a **relativistic jet** at large distances from the black hole.

7.4 Practice: Exercises and Answers

Exercise 7.1. Evaluate the Alfvén speed in the solar corona above a large sunspot.

Answer.

From definition we find

$$V_{\rm A} \approx 2.18 \times 10^{11} \, \frac{B}{\sqrt{n}} \, , \ \ {\rm cm\,s^{-1}} \, . \tag{7.33}$$

Above a sunspot $B \approx 3000$ G, $n \approx 2 \times 10^{8}\,{\rm cm^{-3}}$.

Thus (7.33) gives **unacceptably high** values:

$$V_{\rm A} \approx 5 \times 10^{10} \ {\rm cm\,s^{-1}} > c \, .$$

This means that

> in a strong magnetic field and low density plasma, the Alfvén waves propagate with velocities approaching the light speed c.

So the non-relativistic formula (7.33) has to be corrected by a **relativistic** factor:

$$V_{\rm A}^{rel} = \frac{B}{\sqrt{4\pi\rho}} \, \frac{1}{\sqrt{1 + B^2/4\pi\rho c^2}} \, , \tag{7.34}$$

which agrees with (7.14) if $B^2 \ll 4\pi\rho c^2$.

Therefore the relativistic Alfvén wave speed is always smaller than the light speed.

For values of the magnetic field and plasma density mentioned above,

$$V_{\rm A}^{rel} \approx 2 \times 10^{10}\,{\rm cm\,s^{-1}}.$$

Exercise 7.2. Discuss properties of the Lorentz force in terms of the Maxwellian stress tensor (6.11).

Exercise 7.3. Show that the **magnetic tension force** is directed to the local centre of curvature.

Exercise 7.4. For the conditions in the corona, used in Exercise 7.1, estimate the parameter γ^2.

Answer.

Substitute $p_0 = 2n_0 k_{\text{B}} T_0$ in definition (7.13):

For the temperature $T_0 \approx 2 \times 10^6\,\text{K}$ and magnetic field $B_0 \approx 3000\,\text{G}$

$$\gamma^2 \sim 10^{-7}\,.$$

Exercise 7.5. By using formula (6.63) for the energy flux in ideal MHD, find the **magnetic energy influx** into a reconnecting current layer.

Answer.

In this simplest approximation, near the layer, the magnetic field $\mathbf{B} \perp \mathbf{v}$.

In formula (6.63) the product $\mathbf{B} \cdot \mathbf{v} = 0$ and the energy flux density

$$\mathbf{G} = \rho\,\mathbf{v}\left(\frac{v^2}{2} + w\right) + \frac{B^2}{4\pi}\,\mathbf{v}\,. \tag{7.35}$$

If the approximation of a **strong field** is satisfied, the last term in (7.35) is dominating, and we find the Poynting vector directed into the current layer

$$\mathbf{G}_{\mathrm{P}} = \frac{B^2}{4\pi}\,\mathbf{v}\,. \tag{7.36}$$

Exercise 7.6. Consider a weakly gravitating, slowly rotating body such as the Sun, with all nonlinear gravitational effects neglected.

Compute the **gravitational** force and **gravitomagnetic** force from the linearized Einstein equations (see Landau and Lifshitz, *Classical Theory of Field*).

Show that, for a time-independent body, these equations are identical to the Maxwell equations:

$$\mathrm{rot}\,\mathbf{g} = 0\,, \quad \mathrm{div}\,\mathbf{g} = -\,4\pi\,G\rho_m\,, \tag{7.37}$$

$$\mathrm{rot}\,\mathbf{H}_{gr} = -\,16\pi\,G\,\rho_m\mathbf{v}\,, \quad \mathrm{div}\,\mathbf{H}_{gr} = 0\,. \tag{7.38}$$

Here **the differences** are:

(a) two minus signs because **gravity is attractive** rather than repulsive,

(b) the factor 4 in the rot $\mathbf{H}_{gr}$ equation,

(c) the presence of the gravitational constant G,

(d) the replacement of charge density ρ^{q} by mass density ρ_m, and

(e) the replacement of electric current $\mathbf{j}$ by the mass flow $\rho_m\mathbf{v}$.

Chapter 8

Plasma Flows in a Strong Magnetic Field

A strong magnetic field **easily moves** a rarified plasma in many non-stationary phenomena in the astrophysical environment.

The best studied example is the **solar flares** which strongly influence the interplanetary and terrestrial space.

8.1 The general formulation of a problem

As was shown above, the set of MHD equations for an **ideal** medium in the approximation of **strong** field and **cold** plasma is characterized only by the small parameter $\varepsilon^2 = v^2/V_{\rm A}^2$:

$$\varepsilon^2 \, \frac{d\mathbf{v}}{dt} = - \, \frac{1}{\rho} \, \mathbf{B} \times \mathrm{rot} \, \mathbf{B} \, , \qquad (8.1)$$

$$\frac{\partial \mathbf{B}}{\partial t} = \text{rot}\,(\mathbf{v} \times \mathbf{B})\,, \tag{8.2}$$

$$\frac{\partial \rho}{\partial t} + \text{div}\,\rho \mathbf{v} = 0\,. \tag{8.3}$$

Let us represent all the unknown quantities in the form

$$f(\mathbf{r}, t) = f^{\,(0)}(\mathbf{r}, t) + \varepsilon^2 f^{\,(1)}(\mathbf{r}, t) + \dots\,.$$

Then we try to find the solution in **three consequent steps**.

(a) To zeroth order with respect to ε^2, the magnetic field is determined by the equation

$$\mathbf{B}^{\,(0)} \times \text{rot}\,\mathbf{B}^{\,(0)} = 0\,. \tag{8.4}$$

This must be supplemented with a boundary condition, which generally depends on time:

$$\mathbf{B}^{\,(0)}\,(\mathbf{r}, t)\,|_{\,S} = \mathbf{f}_1\,(\mathbf{r}, t)\,. \tag{8.5}$$

Here S is the boundary of the region G (Fig. 8.1), in which the **force-free-field** Equation (8.4) applies.

> The strong force-free field, changing in time according to the boundary condition (8.5), sets the plasma in motion.

(b) **Kinematics** of this motion is uniquely determined by two conditions.

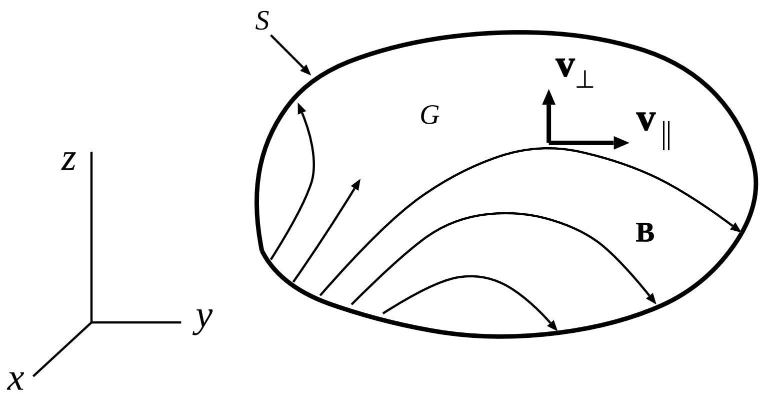


Figure 8.1: The boundary and initial conditions for the ideal MHD problems.

The first one signifies the orthogonality of acceleration to the magnetic field lines

$$\mathbf{B}^{(0)} \cdot \frac{d\mathbf{v}^{(0)}}{dt} = 0\,. \tag{8.6}$$

This equation is the scalar product of Equation (8.1) and the vector $\mathbf{B}^{(0)}$.

The second condition is a consequence of the **freezing-in** Equation (8.2)

$$\frac{\partial\,\mathbf{B}^{(0)}}{\partial t} = \text{rot}\left(\mathbf{v}^{(0)} \times \mathbf{B}^{(0)}\right). \tag{8.7}$$

Equations (8.6) and (8.7) determine the velocity $\mathbf{v}^{(0)}(\mathbf{r}, t)$, if the **initial condition** inside the region G is given:

$$\mathbf{v}_{\parallel}^{(0)}(\mathbf{r}, 0)\,|_{\,G} = \mathbf{f}_2\,(\mathbf{r})\,. \tag{8.8}$$

Here $\mathbf{v}_{\parallel}^{(0)}$ is the velocity component along the field lines.

The velocity component across the field lines, $\mathbf{v}_{\perp}^{(0)}$, is uniquely defined by the freezing-in Equation (8.7) at any moment, including the initial one.

Therefore we do **not** need the initial condition for $\mathbf{v}_{\perp}^{(0)}$.

(c) Since we know the velocity field $\mathbf{v}^{(0)}(\mathbf{r}, t)$, the continuity equation

$$\frac{\partial \rho^{(0)}}{\partial t} + \operatorname{div} \rho^{(0)} \mathbf{v}^{(0)} = 0 \tag{8.9}$$

allows us to find the **plasma density** $\rho^{(0)}(\mathbf{r}, t)$, if we know its initial distribution

$$\rho^{(0)}(\mathbf{r}, 0) \,|_G = f_3(\mathbf{r})\,. \tag{8.10}$$

Therefore,

at any moment of time,

the field $\mathbf{B}^{(0)}(\mathbf{r}, t)$ is found from Equation (8.4) and the boundary condition (8.5).

Thereupon the velocity $\mathbf{v}^{(0)}(\mathbf{r}, t)$ is determined from Equations (8.6) and (8.7) and the initial condition (8.8).

Finally the continuity Equation (8.9) and the initial condition (8.10) give the plasma density $\rho^{(0)}(\mathbf{r}, t)$.

We restrict our attention to the **zeroth order** relative to the parameter ε^2, neglecting the field deviation from a **force-free** state.

The question of the **existence of solutions** will be considered later on, using 2D problems.

8.2 The formalism of 2D problems

Being relatively simple from the mathematical viewpoint, **2D ideal MHD** problems allow us to gain some general knowledge concerning the actual flows of plasma with the frozen-in strong magnetic field.

The 2D problems are sometimes a close approximation of the real 3D flows and can be used to compare the theory with experiments and observations.

There are **two types of problems** treating the plane flows of plasma, i.e. the flows with the velocity field

$$\mathbf{v} = \{\, v_x(x, y, t),\; v_y(x, y, t),\; 0 \,\}\,.$$

All the quantities are dependent on variables x, y and t.

8.2.1 The first type of problems

The first type incorporates the problems with a magnetic field which is everywhere parallel to the z axis:

$$\mathbf{B} = \{\, 0,\; 0,\; B\,(x, y, t) \,\}\,.$$

The corresponding current is parallel to the (x, y) plane:

$$\mathbf{j} = \{\, j_x(x, y, t),\; j_y(x, y, t),\; 0 \,\}\,.$$

As an example, let us discuss the effect of a **longitudinal** magnetic field in a reconnecting current layer (**RCL**).

Under real conditions, reconnection does occur not at the zeroth lines but rather at the **separators**.

The latter differ from the zeroth lines only in that the separators contain the longitudinal field (Fig. 8.2).

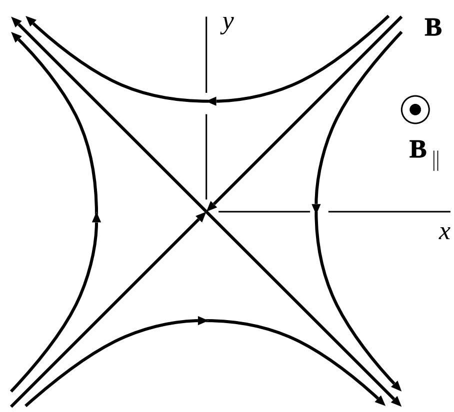


Figure 8.2: A longitudinal field $\mathbf{B}_{\parallel}$ parallel to the z axis is superimposed on the 2D hyperbolic field in the plane (x, y).

With appearance of the longitudinal field, the **force balance** in the RCL is changed.

The field and plasma pressure **outside** the RCL must balance not only the gas pressure but also that of the longitudinal field **inside** the RCL (Fig. 8.3)

$$\mathbf{B}_{\parallel} = \left\{ 0,\ 0,\ B_{\parallel}\,(x, y, t) \right\}.$$

If the longitudinal field accumulated in the RCL during reconnection, the field pressure $B_{\parallel}^2/8\pi$ **would** considerably limit the layer compression and the reconnection rate.

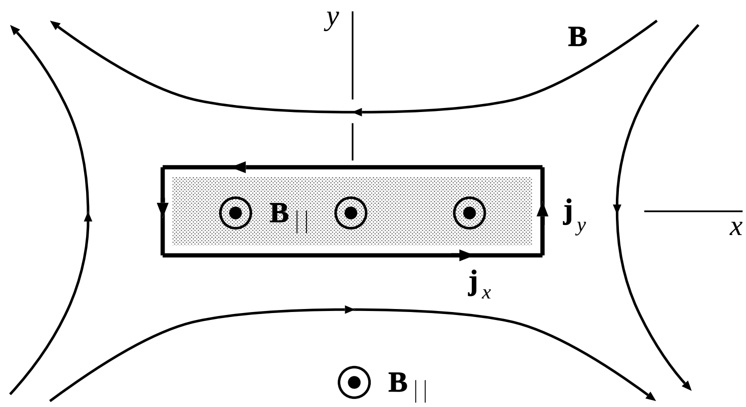


Figure 8.3: A model of a RCL with a longitudinal component $\mathbf{B}_{\parallel}$ of magnetic field.

However the solution of the problem of the **first type** with respect to $\mathbf{B}_{\parallel}$ shows that another effect is of importance in the real plasma with **finite conductivity**.

The **longitudinal field compression** in the RCL produces a gradient of this field and a corresponding **current circulating** in the transversal plane (x, y).

This current is represented schematically in Fig. 8.3.

Dissipation of the circulating current leads to longitudinal field diffusion outwards from the RCL.

More exactly, because of dissipation, **plasma moves** into the RCL relatively free with respect to the longitudinal component of magnetic field, thus limiting its accumulation in the RCL.

8.2.2 The second type of MHD problems

8.2.2 (a) Magnetic field and its vector potential

The 2D problems of the second type treat the **plane** flows

$$\mathbf{v} = \{\, v_x(x, y, t),\; v_y(x, y, t),\; 0 \,\}\,,$$

associated with the **plane** magnetic field

$$\mathbf{B} = \{\, B_x(x, y, t),\; B_y(x, y, t),\; 0 \,\}\,.$$

The currents corresponding to this field are parallel to the z axis

$$\mathbf{j} = \{\, 0,\; 0,\; j\,(x, y, t) \,\}\,.$$

The vector-potential **A** has an its only non-zero component:

$$\mathbf{A} = \{\, 0,\; 0,\; A\,(x, y, t) \,\}\,.$$

The magnetic field **B** is defined as

$$\mathbf{B} = \left\{ \frac{\partial A}{\partial y}\,,\; -\frac{\partial A}{\partial x}\,,\; 0 \right\}. \tag{8.11}$$

The scalar function $A\,(x, y, t)$ is often termed the **vector potential**.

This function is quite useful, owing to its properties.

Property 1.

Substitute (8.11) in the differential equations describing the magnetic field lines

$$\frac{dx}{B_x} = \frac{dy}{B_y} = \frac{dz}{B_z}\,.$$

These equations imply parallelism of the vector

$$d\,\mathbf{l} = \{dx,\; dy,\; dz\}$$

to the vector $\mathbf{B} = \{B_x, B_y, B_z\}$.

In the case under study

$$B_z = 0,\;\; dz = 0\,,$$

and

$$\frac{dx}{\partial A/\partial y} = -\frac{dy}{\partial A/\partial x}$$

or

$$\frac{\partial A}{\partial x}\,dx + \frac{\partial A}{\partial y}\,dy = 0\,.$$

On integrating the last, we come to the relation

$$\boxed{A\,(x, y, t) = \text{const} \quad \text{for} \quad t = \text{const}\,.} \tag{8.12}$$

This is the equation for a family of **magnetic field lines** in the plane $z = \text{const}$ at the moment t.

Property 2.

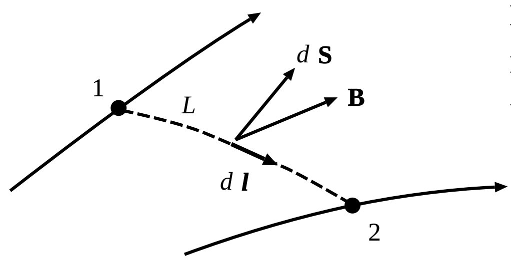


Figure 8.4: The curve L connects the points 1 and 2 situated in different field lines.

Let L be a curve in the plane (x, y) and $d\,\mathbf{l}$ an arc element along this curve (Fig. 8.4).

Let us calculate the magnetic flux $d\,\Phi$ through the arc element $d\,\mathbf{l}$.

By definition,

$$d\,\Phi = \mathbf{B}\cdot d\,\mathbf{S} = \mathbf{B}\cdot(\mathbf{e}_z \times d\,\mathbf{l}\,) = \mathbf{B}\cdot \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ 0 & 0 & 1 \\ dx & dy & 0 \end{vmatrix} =$$

$$= \mathbf{B}\cdot\{\,(-dy)\,\mathbf{e}_x + dx\,\mathbf{e}_y\,\} =$$

$$= -B_x\,dy + B_y\,dx\,. \tag{8.13}$$

On substituting (8.11) in (8.13) we find

$$d\,\Phi = -\frac{\partial A}{\partial y}\,dy - \frac{\partial A}{\partial x}\,dx = -\,dA\,.$$

On integrating this along the curve L from point 1 to point 2 we obtain the magnetic flux

$$\Phi = A_2 - A_1\,.$$

Thus the fixed value of the potential A is not only the field line 'tag' determined by formula (8.12);

> the difference of values of the vector potential A on two field lines is equal to the **magnetic flux** between them.

Simple rule:

Plot the field lines corresponding to equidistant values of A.

Property 3.

Let us substitute definition (8.11) in the freezing-in equation.

We obtain the following equation

$$\text{rot}\,\frac{d\mathbf{A}}{dt} = 0\,.$$

Disregarding a gradient of an arbitrary function and considering the second type of 2D problems, we have

$$\frac{dA}{dt} \equiv \frac{\partial A}{\partial t} + (\mathbf{v} \cdot \nabla)A = 0\,. \tag{8.14}$$

This equation means that the lines

$$A\,(x, y, t) = \text{const} \tag{8.15}$$

are **Lagrangian** lines: they move together with plasma.

According to (8.12) they are composed of the field lines.

Hence Equation (8.14) expresses the magnetic field **freezing in** plasma.

Thus we have one of the integrals of motion

$$\boxed{A\,(x, y, t) = A\,(x_0, y_0, 0) \equiv A_0} \tag{8.16}$$

at an arbitrary t.

Here

x_0, y_0 are the coordinates of a "**fluid particle**" at the initial time $t = 0$;

x, y are the coordinates of the **same** particle at a moment of time t or the coordinates of **any other** particle situated on the same field line A_0 at the moment t.

Property 4.

Equation of motion (8.1) rewritten in terms of the vector potential $A(x, y, t)$ is of the form

$$\varepsilon^2 \, \frac{d\mathbf{v}}{dt} = -\frac{1}{\rho} \, \Delta A \, \nabla A \, . \tag{8.17}$$

In the zeroth order relative to the small parameter ε^2, outside the **zeroth points** (where $\nabla A = 0$) and the magnetic **field sources** (where $\Delta A \neq 0$) we have:

$$\boxed{\Delta A = 0 \, .} \tag{8.18}$$

So the vector potential is a **harmonic** function of variables x and y.

Hence, considering the (x, y) plane as a complex plane

$$z = x + \mathrm{i}\, y\,,$$

it is convenient to relate an **analytic** function F to the vector potential A:

$$F\,(z, t) = A\,(x, y, t) + \mathrm{i}\, A^{+}(x, y, t)\,. \tag{8.19}$$

Here $A^{+}(x, y, t)$ is a **conjugate harmonic** function connected with $A\,(x, y, t)$ by the Cauchy-Riemann condition

$$A^{+}(x, y, t) = \int \left(- \frac{\partial A}{\partial y}\, dx + \frac{\partial A}{\partial x}\, dy \right) + A^{+}(t) =$$

$$= - \int \mathbf{B} \cdot d\,\mathbf{l} + A^{+}(t)\,,$$

where $A^{+}(t)$ is a quantity independent of the coordinates x and y.

The function $F(z, t)$ is termed the **complex potential**. The magnetic field vector

$$\mathbf{B} = B_x + \mathrm{i}\, B_y = -\,\mathrm{i} \left(\frac{dF}{dz} \right)^{*}, \tag{8.20}$$

the asterisk denoting the complex conjugation.

Now we can apply the methods of the complex variable function theory, in particular the method of **conform mapping** .

This has been done in order to determine the **structure of magnetic field**:

- in vicinity of **reconnecting current layer** (Syrovatskii, 1971),
- in solar **coronal streamers** (Somov and Syrovatskii, 1972)
- in the Earth's **magnetosphere** (Oberz, 1973),
- the **accretion disk** magnetosphere (Somov et al., 2003).

Markovskii and Somov (1989) generalized the Syrovatskii model by **attaching** four **shock MHD waves** at the edges of the RCL.

The model reduces to the Riemann-Hilbert problem (in an analytical form on the basis of the Christoffel-Schwarz integral) in order to analyze the structure of magnetic field in vicinity of reconnection region (Bezrodnykh et al., 2007).

8.2.2 (b) Motion of the plasma and its density

The motion kinematics due to changes in a potential field is uniquely determined by **two conditions**:

(i) the freezing-in condition and

(ii) the acceleration orthogonality with respect to the field lines

$$\frac{d\mathbf{v}^{(0)}}{dt} \times \nabla A^{(0)} = 0\,. \tag{8.21}$$

Equation (8.21) is a result of eliminating the unknown $\Delta A^{(1)}$ from two components of the vector equation

$$\frac{d\mathbf{v}^{(0)}}{dt} = -\,\frac{1}{\rho^{(0)}}\,\Delta A^{(1)}\,\nabla A^{(0)}\,. \tag{8.22}$$

If $x(t)$ and $y(t)$ are the coordinates of a **fluid particle**, Equations (8.21) and (8.14) are reduced to the **ordinary differential** equations (Somov and Syrovatskii, 1976).

Once the kinematic part of the problem is solved, the **trajectories** of fluid particles are known:

$$x = x\,(x_0\,,\,y_0\,,t)\,, \qquad y = y\,(x_0\,,\,y_0\,,t)\,. \tag{8.23}$$

The fluid particle **density change** on moving along its trajectory is determined by the continuity Equation (8.3), rewritten in the Lagrangian form, and is equal to

$$\frac{\rho\,(x,\,y,\,t)}{\rho_0\,(x_0,\,y_0)} = \frac{d\,U_0}{d\,U} = \frac{\mathcal{D}\,(x_0,\,y_0)}{\mathcal{D}\,(x,\,y)}\,. \tag{8.24}$$

Here $d\,U_0$ is the initial volume of a particle, $d\,U$ is the volume of the same particle at a moment of time t;

$$\frac{\mathcal{D}\,(x_0,\,y_0)}{\mathcal{D}\,(x,\,y)} = \frac{\partial x_0}{\partial x}\,\frac{\partial y_0}{\partial y} - \frac{\partial x_0}{\partial y}\,\frac{\partial y_0}{\partial x} \tag{8.25}$$

is the Jacobian of the transformation that is inverse to the transformation (8.23).

The 2D equations of the strong-field-cold-plasma approximation are relatively **simple but useful** for applications to astrophysical plasmas.

In particular, they enable us to study the fast plasma flows in the solar atmosphere and to understand some aspects of the reconnection process.

> In spite of their numerous applications, **the list** of exact solutions to them is rather **poor**. Still, we can enrich it significantly,

relying on many astrophysical objects, for example in the **accretion disk coronae** and some mathematical ideas.

8.3 The existence of continuous flows

Thus, in the strong-field-cold-plasma approximation, the MHD equations for a plane 2D flow of ideally conducting plasma (for the second-type problems) are reduced, in the zeroth order in the small parameter ε^2, to the following **closed set** of equations:

$$\Delta A = 0 \,, \tag{8.26}$$

$$\frac{d\mathbf{v}}{dt} \times \nabla A = 0 \,, \tag{8.27}$$

$$\frac{dA}{dt} = 0 \,, \tag{8.28}$$

$$\frac{\partial \rho}{\partial t} + \operatorname{div} \rho \mathbf{v} = 0\,. \tag{8.29}$$

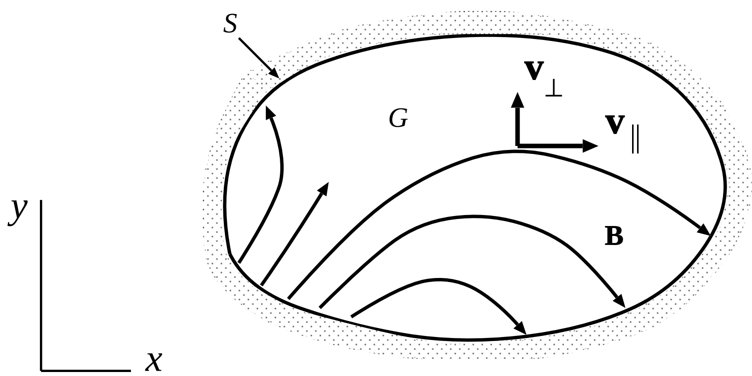


Figure 8.5: The boundary and initial conditions for a second-type 2D problem.

The solution of this set is completely defined inside some region G (Fig. 8.5) once the **boundary condition** is given at the boundary S

$$A\,(x, y, t)\,|_S = f_1\,(x, y, t) \tag{8.30}$$

together with the **initial conditions** inside the region G

$$\mathbf{v}_\parallel\,(x, y, 0)\,|_G = \mathbf{f}_2\,(x, y)\,, \tag{8.31}$$

$$\rho\,(x, y, 0)\,|_G = f_3\,(x, y)\,. \tag{8.32}$$

Here $\mathbf{v}_\parallel$ is the velocity component along field lines.

Once the potential $A\,(x, y, t)$ is known, the transversal velocity component is uniquely determined by the freezing-in Equation (8.28) and is equal, at any moment **including the initial one**, to

$$\mathbf{v}_{\perp}(x, y, t) = (\mathbf{v} \cdot \nabla A)\,\frac{\nabla A}{|\,\nabla A\,|^2} = -\frac{\partial A}{\partial t}\,\frac{\nabla A}{|\,\nabla A\,|^2}\,. \qquad (8.33)$$

The density $\rho\,(x, y, t)$ is found from the continuity Equation (8.29) and the initial density distribution (8.32).

The next Section is devoted to an example which may have applications.

8.4 Flows in a time-dependent dipole field

8.4.1 Plane magnetic dipole fields

Two parallel currents, equal in magnitude but opposite in direction, engender the magnetic field which far from the currents can be described by a complex potential

$$F\,(z) = \frac{\mathrm{i}\,\mathbf{m}}{z}\,, \qquad \mathbf{m} = m\,e^{\,\mathrm{i}\psi} \qquad (8.34)$$

and is called the plane **dipole** field.

The quantity

$$m = \frac{2}{c}\,I\,l$$

has the meaning of the **dipole moment**,

I is the current magnitude,
l is the distance between the currents.

Formula (8.34) corresponds to the dipole situated at the origin of coordinates in the plane (x, y) and directed at an angle of ψ to the x axis.

Let us consider the **plasma flow** caused by the change with time of the strong magnetic field of the dipole
$\psi = \pi/2$ and
$m = m(t)\,,\ m(0) = m_0\,.$

(a) Let us find the **first integral** of motion.

According to(8.34), the complex potential

$$F\,(z, t) = \frac{-\,m(t)\,x + \mathrm{i}\,m(t)\,y}{x^2 + y^2}\,. \tag{8.35}$$

So, the **field lines** constitute a family of circles

$$A\,(x, y, t) = -\,\frac{m(t)\,x}{x^2 + y^2} = \text{const} \quad \text{for} \quad t = \text{const}\,.$$

They have centres on the axis x and the common point $x = 0,\ y = 0$ in Fig. 8.6.

Therefore the **freezing-in condition** (8.16) results in a first integral of motion

$$\frac{m\,x}{x^2 + y^2} = \frac{m_0\,x_0}{x_0^2 + y_0^2}\,. \tag{8.36}$$

Here x_0, y_0 are the coordinates of a fluid particle at the initial time $t = 0$.

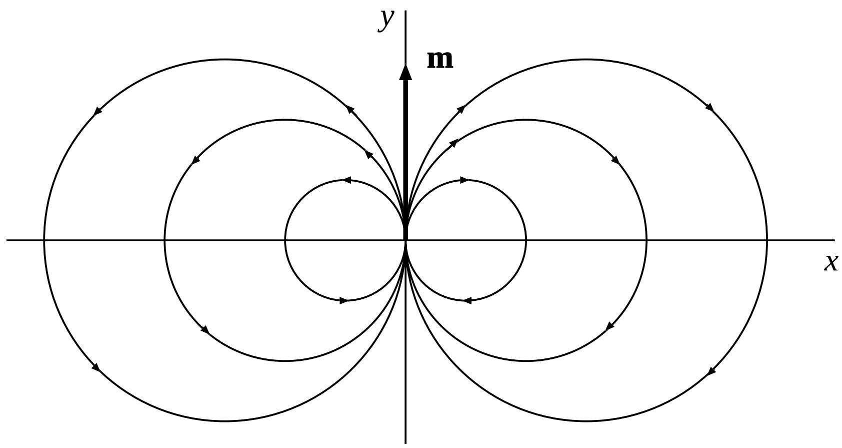


Figure 8.6: The field lines of a plane magnetic dipole.

(b) The **second integral** is easily found in the limit of small changes of the dipole moment $m\,(t)$ and respectively **small** plasma displacements.

Assuming the parameter

$$\delta = v\tau / L$$

to be small, Equation (7.26):

$$\varepsilon^2 \frac{\partial \mathbf{v}}{\partial t} = -\frac{1}{\rho}\,\mathbf{B} \times \mathrm{rot}\,\mathbf{B}\,,$$

which is **linear in velocity**.

The integration over time (with **zero initial values** for the velocity) allows us to reduce Equation (7.26) to the form

$$\frac{\partial x}{\partial t} = K(x, y, t)\,\frac{\partial A}{\partial x}\,, \qquad \frac{\partial y}{\partial t} = K(x, y, t)\,\frac{\partial A}{\partial y}\,. \qquad (8.37)$$

Here $K(x, y, t)$ is some function of coordinates and time.

Eliminating it from two Equations (8.37), we arrive at

$$\frac{\partial y}{\partial x} = \frac{\partial A}{\partial y} \bigg/ \frac{\partial A}{\partial x} \,. \tag{8.38}$$

Thus, not only the acceleration but also the plasma **displacements are normal to the field lines**.

For dipole field, we obtain an ordinary differential equation

$$\frac{dy}{dx} = \frac{2xy}{x^2 - y^2} \,.$$

Its integral

$$\frac{y}{x^2 + y^2} = \text{const}$$

describes **a family of circles, orthogonal to the field lines**, and presents fluid particle trajectories.

In particular, the trajectory of a particle, situated at a point (x_0, y_0) at the initial time $t = 0$, is an **arc of the circle**

$$\frac{y}{x^2 + y^2} = \frac{y_0}{x_0^2 + y_0^2} \tag{8.39}$$

from the point (x_0, y_0) to the point (x, y) on the field line (8.36) (Fig. 8.7).

The integrals of motion (8.36) and (8.39) completely determine the plasma flow in terms of the Lagrangian coordinates

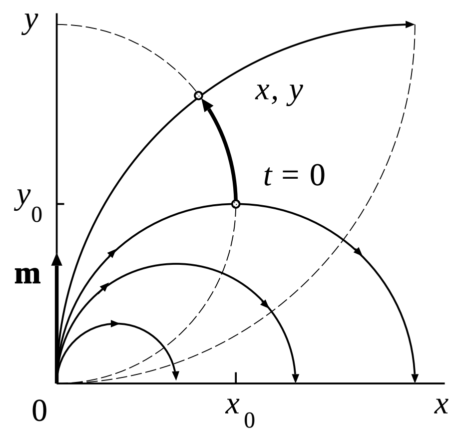


Figure 8.7: A **trajectory** of a fluid particle driven by a changing magnetic field of a plane dipole.

$$x = x\,(x_0, y_0, t)\,, \qquad y = y\,(x_0, y_0, t)\,. \tag{8.40}$$

This flow has a **simple form**:

the particles are connected with the magnetic field lines and move together with them in a transversal direction.

This is a result of considering **small** displacements under the action of the force perpendicular to the field lines.

(c) The **plasma density** change.

On calculating the Jacobian for the transformation given by (8.36) and (8.39), we obtain the formula

$$\frac{\rho\,(x, y, t)}{\rho_0} = \left(\frac{m}{m_0}\right) \frac{m_0^4}{(m^2 x^2 + m_0^2\, y^2)^4}\, \Big\{ \big[\, m^2 x^4 + m_0^2\, y^4 +$$

$$+ x^2 y^2 \left(3m^2 - m_0^2\right) \big]^2 - \big[\, 2x^2 y^2 \left(m_0^2 - m^2\right) \big]^2 \Big\}\,. \tag{8.41}$$

On the dipole axis ($x = 0$)

$$\boxed{\frac{\rho\,(0, y, t)}{\rho_0} = \frac{m}{m_0}\,,} \tag{8.42}$$

whereas in the 'equatorial plane' ($y = 0$)

$$\frac{\rho\,(x, 0, t)}{\rho_0} = \left(\frac{m_0}{m}\right)^3. \tag{8.43}$$

> With increasing dipole moment m, the plasma density on the dipole axis grows proportionally to the moment,

whereas that at the **equatorial plane** falls in inverse proportion to the third power of the moment.

The result pertains to the small changes in the dipole moment.

The exception is formula (8.42).

It applies to **any changes** of the dipole moment.

The acceleration of plasma is perpendicular to the field lines and is zero at the dipole axis.

Hence, if the plasma is motionless at the initial moment, arbitrary changes of the dipole moment do not cause a plasma motion on the dipole axis ($\mathbf{v} = 0$).

Plasma displacements in the vicinity of the dipole axis always remain small ($\delta \ll 1$) and the solution obtained applies.

(d) In the general case of arbitrarily large dipole moment changes,

> the **inertial effects** resulting in plasma flows along the field lines are of considerable importance

(Somov and Syrovatskii, 1972).

The solution of the problem requires the integration of the **ordinary differential equations** that follows from Equation (8.21) and the freezing-in Equation (8.14).

8.4.2 Axial-symmetric dipole fields

2D axial-symmetric problems can better suit the astrophysical applications of the **second-type problem** considered.

The **ideal** MHD equations are written, using the approximation of a **strong** field and **cold** plasma, in spherical coordinates with due regard for axial symmetry.

The role of the vector potential is fulfilled by the so-called **stream function**

$$\Phi\,(r,\theta,t) = r\sin\theta\, A_{\varphi}(r,\theta,t)\,. \tag{8.44}$$

Here A_{φ} is the only non-zero φ-component of the vector-potential $\mathbf{A}$.

In terms of the stream functions, the equations take the form

$$\frac{d\mathbf{v}}{dt} = \varepsilon^{-2} K(r,\theta,t)\,\nabla\Phi\,, \qquad \frac{d\,\Phi}{dt} = 0\,, \qquad \frac{d\rho}{dt} = -\rho\,\mathrm{div}\,\mathbf{v}\,,$$

where

$$K(r,\theta,t) = \frac{j_\varphi(r,\theta,t)}{\rho\, r \sin\theta} \tag{8.45}$$

(Somov and Syrovatskii, 1976).

The equations formally coincide with the corresponding Equations (8.17), (8.14) and (8.3) describing the plane flows in terms of the vector potential.

As a zeroth approximation in the small parameter ε^2, we may take, for example, the **dipole field**.

In this case the stream function is of the form

$$\Phi^{(0)}(r,\theta,t) = m(t)\, \frac{\sin^2\theta}{r}\,, \tag{8.46}$$

where $m(t)$ is a time-varying moment.

Let us imagine a **magnetized ball** of radius $R(t)$ with the frozen field $\mathbf{B}_{int}(t)$.

The dipole moment of such a ball (a star or its envelope)

$$m(t) = \frac{1}{2}\, B_{int}(t)\, R^3(t) = \frac{1}{2\pi}\left(B_0\, \pi R_0^2\right) R(t)\,. \tag{8.47}$$

Here B_0 and R_0 are the values of $B_{int}(t)$ and $R(t)$ at the initial time $t = 0$.

The second equality takes account of conservation of the flux

$$B_{int}(t)\, \pi R^2(t)$$

through the ball.

Formula (8.47) shows that the dipole moment of the ball is proportional to its radius $R(t)$.

The solution to the problem (Somov and Syrovatskii, 1972a) shows that as the dipole moment grows

> the magnetic field **rakes** the plasma **up** to the dipole axis, **compresses** it and simultaneously **accelerates** it along the field lines.

The **density at the axis** grows in proportion to the dipole moment, just as in the 2D plane case.

* * *

Envelopes of **nova** and **supernova** stars present a wide variety of different shapes.

It is common to find either **flattened** or **stretched** envelopes.

Their surface brightness is maximal at the ends of the main axes of an oval image.

This can sometimes be interpreted as a **gaseous ring** observed almost from an edge.

However, if there is no luminous belt between the brightness maxima, then the remaining possibility is that single gaseous compressions – **condensations** – exist in the envelope.

At the early stages of the expansion, they give the impression that the nova '**bifurcates**'.

Suppose that the star's magnetic field was a dipole one before the explosion.

At the moment of the explosion a **massive envelope** with the frozen-in field separated from the star and began to expand.

The expansion results in the growth of the dipole moment.

The field **rakes** the interstellar plasma surrounding the envelope, as well as **external layers** of the envelope, up in the direction of the dipole axis.

The process can be divided into **two stages**.

At the first one, the plasma is raked up by the field into the polar regions, a growth in density and pressure at the dipole axis taking place.

At the second stage, the increased pressure hinders the growth of the density, thus stopping compression, but the raking-up still continues.

The gas **pressure gradient**, arising ahead of the envelope, gives rise to the motion along the axis.

As a result, all the plasma is raked up into **two compact condensates**.

* * *

If a magnetized ball **compresses**, plasma flows from the poles to the equatorial plane, thus forming a **dense disk** or **ring**.

This is the old problem of astrophysics concerning the compression of a gravitating **cloud with a frozen-in field**.

Magnetic raking-up of plasma into dense disks can work in the atmospheres of **collapsing stars**.

8.5 Practice

Exercise 8.1. For a 3D field **B**, consider properties of the vector-potential **A** which is determined in terms of two scalar functions α and β:

$$\mathbf{A} = \alpha \nabla \beta + \nabla \psi \,. \qquad (8.48)$$

Here ψ is an arbitrary scalar function.

Answer.

Formula (8.48) permits **B** to be written as

$$\mathbf{B} = \nabla \alpha \times \nabla \beta \,. \qquad (8.49)$$

Hence

$$\mathbf{B} \cdot \nabla \alpha = 0 \quad \text{and} \quad \mathbf{B} \cdot \nabla \beta = 0 \,. \qquad (8.50)$$

Thus $\nabla \alpha$ and $\nabla \beta$ are perpendicular to the vector **B**, and functions α and β are constant along **B**.

The surfaces $\alpha = \text{const}$ and $\beta = \text{const}$ are orthogonal to their gradients and tangent to **B**.

Hence

> a magnetic field line can be conveniently defined in terms of a pair of values: α and β.

The functions α and β are referred to as **Euler potentials** or **Clebsch variables**.

Advantage of these variables appears in the study of field line motions.

Exercise 8.2. Evaluate the typical value of the dipole moment for a **neutron star**.

Answer.

Typical neutron stars have $B \sim 10^{12}$ G.

With the star radius $R \sim 10$ km, it follows from formula (8.47) that

$$m \sim 10^{30}\,\text{G cm}^3\,.$$

Some of neutron stars are the spinning super-magnetized neutron stars created by **supernova** explosions.

The rotation of such stars called **magnetars** is slowing down so rapidly that a **super-strong** field,

$$B \sim 10^{15}\ \text{G}\,,$$

could provide so fast braking.

For a magnetar

$$m \sim 10^{33}\,\text{G cm}^3\,.$$

Exercise 8.3. Show that, prior to a solar flare, the magnetic energy density in the corona is of about three orders of magnitude greater than any of the other types.

Exercise 8.4. By using the method of **conform mapping**, determine the shape of a magnetic cavity, **magnetosphere**, created by a plane dipole inside a perfectly conducting uniform plasma with a gas pressure p_0.

Answer.

The conditions to be satisfied along the boundary S of the magnetic cavity G are equality of magnetic and gas pressure,

$$\left. \frac{B^2}{8\pi} \right|_S = p_0 = \text{const}\,, \tag{8.51}$$

and tangency of the magnetic field,

$$\mathbf{B} \cdot \mathbf{n} \,\big|_S = 0\,. \tag{8.52}$$

Condition (8.52) means that

$$\text{Re}\, F(z) = A\,(x, y) = \text{const}\,, \tag{8.53}$$

where a complex potential $F(z)$ is an analytic function within the region G except at the point $z = 0$ of the dipole $\mathbf{m}$.

Let us assume that a conform transformation $w = w(z)$ maps the region G onto the circle

$$|\,w\,| \leq 1$$

in an auxiliary complex plane

$$w = u + \text{i}v$$

so that the point $z = 0$ goes into the centre of the circle (Fig 8.8).

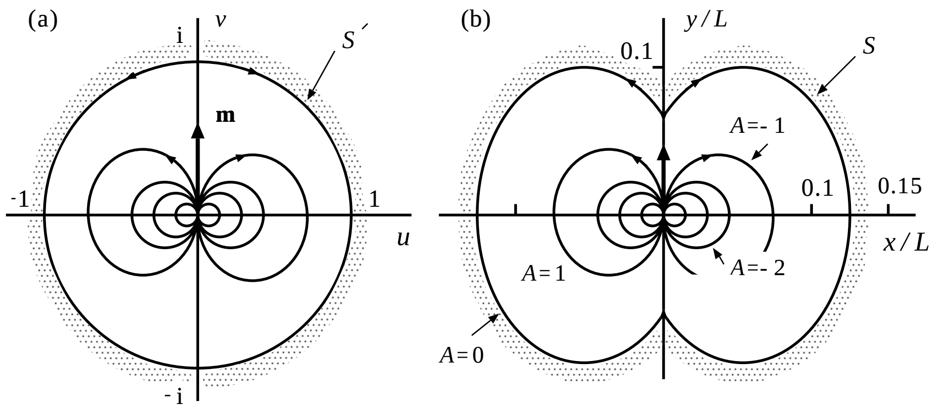


Figure 8.8: The field lines of a dipole **m** inside: (a) the unit circle in the plane w, (b) the cavity in a plasma of constant pressure.

The boundary $|\,w\,| = 1$ is the field line S' of the solution in the plane w, which we construct:

$$F(w) = \left(w - \frac{1}{w} \right). \tag{8.54}$$

Note that we have used only the boundary condition (8.52).

The other boundary condition (8.51) allows us to find an unknown transformation $w = w(z)$.

The field lines are shown in Fig. 8.8b.

This solution can be used in the zero-order approximation to analyze properties of plasma flows near collapsing or exploding astrophysical objects with strong magnetic fields.

Exercise 8.5 To estimate a large-scale magnetic field in the corona of an accretion disk, we have to find the structure of the field inside an open magnetosphere created by a dipole field of a star and a regular field generated by the disk (Somov et al., 2003).

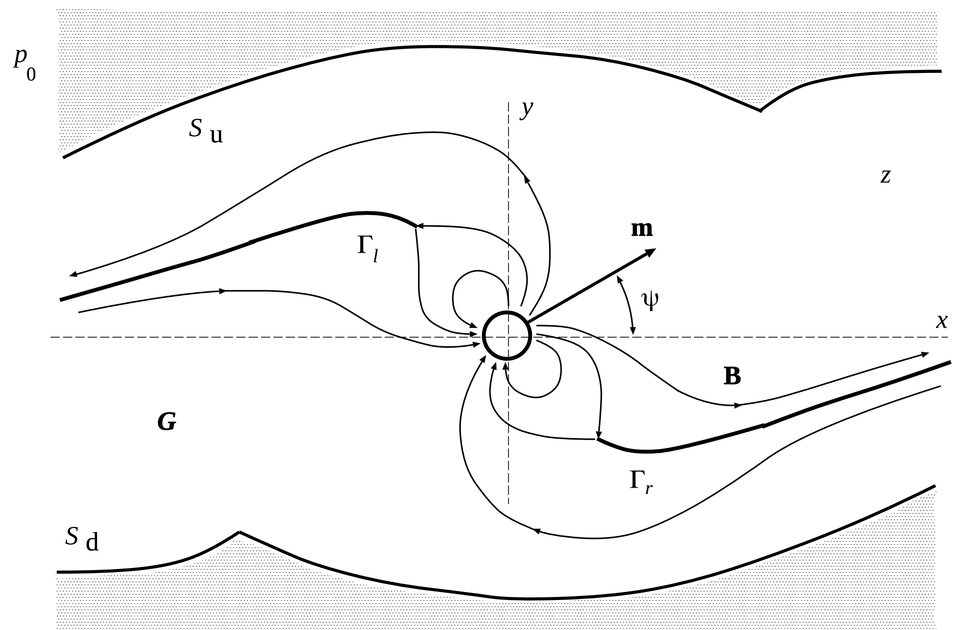


Figure 8.9: A model of the star magnetosphere with an accretion disk; Γ_l and Γ_r are the cross sections of the disk. $S_{\rm u}$ and $S_{\rm d}$ together with Γ_l and Γ_r constitute the boundary of the domain G in the plane z.

Consider a 2D problem, demonstrated by Fig. 8.9, on the shape of a magnetic cavity and the shape of the accretion disk under assumption that this cavity, i.e. the magnetosphere, is surrounded by a perfectly conducting uniform plasma with a gas pressure p_0.

Discuss a way to solve the problem by using the method of conform mapping.

Chapter 9

MHD Waves in Astrophysical Plasma

There are four different modes of MHD waves in an ideal plasma with magnetic field.

They can create turbulence, accelerate particles and produce a lot of interesting effects in astrophysical plasmas.

9.1 The dispersion equation in ideal MHD

Small disturbances in a conducting medium with a magnetic field propagate as **waves**, their properties being **different from** those of the sound waves in a **gas** or electromagnetic waves in a **vacuum**.

First, the conducting medium with a magnetic field has a characteristic **anisotropy**: the wave propagation velocity depends upon the direction of propagation relative to the field.

Second, as a result of the interplay of electromagnetic and hydrodynamic phenomena, the waves in MHD are generally **neither** longitudinal **nor** transversal.

The study of the small-amplitude waves, apart from being interesting in itself, has a direct bearing on the analysis of **large-amplitude waves**, in particular **shock waves** and other discontinuous flows, including **reconnecting current layers**.

Initially we shall study the possible types of small-amplitude waves in **ideal** MHD.

Suppose a plasma in the initial **stationary** state is subjected to a small perturbation: velocity $\mathbf{v}_0$, field $\mathbf{B}_0$, density ρ_0, pressure p_0 and entropy s_0 acquire some small deviations $\mathbf{v}\,'$, $\mathbf{B}\,'$, $\rho\,'$, $p\,'$ and $s\,'$:

$$\begin{array}{lll} \mathbf{v} = \mathbf{v}_0 + \mathbf{v}\,' , & \mathbf{B} = \mathbf{B}_0 + \mathbf{B}\,' , & \\ \rho = \rho_0 + \rho\,' , & p = p_0 + p\,' , & s = s_0 + s\,' . \end{array} \tag{9.1}$$

The initial state is assumed to be a **uniform** flow of an **homogeneous** medium in a **constant** magnetic field:

$$\begin{array}{lll} \mathbf{v}_0 = \text{const} , & \mathbf{B}_0 = \text{const} , & \\ \rho_0 = \text{const} , & p_0 = \text{const} , & s_0 = \text{const} . \end{array} \tag{9.2}$$

The latter simplification can be ignored, i.e. we may study waves in **inhomogeneous** media, the coefficients in linearized equations being dependent upon the coordinates.

For the sake of simplicity we restrict our consideration to the case (9.2).

It is convenient to introduce the following designations:

$$\mathbf{u} = \frac{\mathbf{B}_0}{\sqrt{4\pi\rho_0}}\,, \qquad \mathbf{u}\,' = \frac{\mathbf{B}\,'}{\sqrt{4\pi\rho_0}}\,. \tag{9.3}$$

Let us linearize the set of MHD equations for an ideal medium.

We substitute (9.1)–(9.3) in Equations (6.56), neglecting the products of small quantities.

Hereafter the subscript '0' will be omitted.

We shall get the following set of **linear differential** equations for the primed quantities characterizing small perturbations:

$$\begin{aligned}
&\partial\,\mathbf{u}\,'/\,\partial t + (\mathbf{v}\cdot\nabla)\,\mathbf{u}\,' = (\mathbf{u}\cdot\nabla)\,\mathbf{v}\,' - \mathbf{u}\,\mathrm{div}\,\mathbf{v}\,',\\
&\mathrm{div}\,\mathbf{u}\,' = 0\,,\\
&\partial\,\mathbf{v}\,'/\,\partial t + (\mathbf{v}\cdot\nabla)\,\mathbf{v}\,' =\\
&= -\,\rho^{-1}\,\nabla\,(\,p\,' + \rho\,\mathbf{u}\cdot\mathbf{u}\,'\,) + (\mathbf{u}\cdot\nabla)\,\mathbf{u}\,',\\
&\partial\rho\,'/\,\partial t\, + (\mathbf{v}\cdot\nabla)\,\rho\,' = -\,\rho\,\mathrm{div}\,\mathbf{v}\,',\\
&\partial s\,'/\,\partial t\, + (\mathbf{v}\cdot\nabla)\,s\,' = 0\,,\\
&p\,' = (\partial p\,/\,\partial\rho)_s\;\rho\,' + (\partial p\,/\,\partial s)_\rho\;s\,'\,.
\end{aligned} \tag{9.4}$$

The latter equation is the linearized **equation of state**.

We rewrite it as follows:

$$p\,' = V_s^{\,2}\rho\,' + b\,s\,'. \tag{9.5}$$

Here

$$V_s = (\partial p \,/\, \partial \rho)_s^{1/2} \tag{9.6}$$

is the velocity of **sound** in a medium without a magnetic field, the coefficient

$$b = (\partial p / \partial s)_\rho \,.$$

The set of Equations (9.4) is that of linear differential equations with **constant coefficients**.

That is why we seek a solution in the form of **plane waves**

$$f\,'(\mathbf{r}, t) \sim \exp\left[\, \mathrm{i}\,(\mathbf{k} \cdot \mathbf{r} - \omega t)\,\right], \tag{9.7}$$

where ω is the wave frequency and $\mathbf{k}$ is the wave vector.

An arbitrary disturbance can be expanded into such waves by means of a Fourier transform.

As this takes place, the set of Equations (9.4) is reduced to the following set of **linear algebraic** equations:

$$\begin{aligned}
&(\omega - \mathbf{k} \cdot \mathbf{v})\,\mathbf{u}' + (\mathbf{k} \cdot \mathbf{u})\,\mathbf{v}' - \mathbf{u}\,(\mathbf{k} \cdot \mathbf{v}') = 0\,,\\
&\mathbf{k} \cdot \mathbf{u}' = 0\,,\\
&(\omega - \mathbf{k} \cdot \mathbf{v})\,\mathbf{v}' + (\mathbf{k} \cdot \mathbf{u})\,\mathbf{u}' - \rho^{-1}\,(\,p' + \rho\,\mathbf{u} \cdot \mathbf{u}'\,)\,\mathbf{k} = 0\,,\\
&(\omega - \mathbf{k} \cdot \mathbf{v})\,\rho' - \rho\,(\mathbf{k} \cdot \mathbf{v}') = 0\,,\\
&(\omega - \mathbf{k} \cdot \mathbf{v})\,s' = 0\,,\\
&p' - V_{\mathrm{s}}^{\,2}\,\rho' - b\,s' = 0\,.
\end{aligned} \tag{9.8}$$

The quantities $\mathbf{k}$ and ω are assumed to be known from the initial conditions.

The unknown terms are the primed ones.

With respect to these the set of Equations (9.8) is closed, linear and **homogeneous** (the right-hand sides equal zero).

For this set to have non-trivial solutions, its determinant must be equal to zero.

It is convenient to use the frequency

$$\omega_0 = \omega - \mathbf{k} \cdot \mathbf{v}\,, \tag{9.9}$$

i.e. the frequency in a frame of reference moving with the plasma.

Setting the determinant equal to zero, we get the following **dispersion** equation

$$\omega_0^2 \left[\omega_0^2 - (\mathbf{k} \cdot \mathbf{u})^2 \right] \times$$
$$\times \left[\omega_0^4 - k^2 \left(V_s{}^2 + u^2 \right) \omega_0^2 + k^2 V_s{}^2 \, (\mathbf{k} \cdot \mathbf{u})^2 \right] = 0\,. \tag{9.10}$$

It defines **four** values of ω_0^2.

Four different modes of waves are defined, each of them having its own velocity of propagation with respect to the plasma

$$\mathbf{V}_{\mathrm{ph}} = \frac{\omega_0}{\mathbf{k}}\,. \tag{9.11}$$

This is the **phase** velocity of a wave.

It is distinguished from the **group** velocity

$$\mathbf{V}_{\mathrm{gr}} = \frac{d\,\omega_0}{d\,\mathbf{k}}\,. \tag{9.12}$$

Let us consider the properties of the waves defined by Equation (9.10).

9.2 Small-amplitude waves in ideal MHD

9.2.1 Entropy waves

The first root of (9.10)

$$\omega_0 = \omega - \mathbf{k} \cdot \mathbf{v} = 0 \tag{9.13}$$

corresponds to the perturbation which is **immobile** with respect to the medium:

$$\mathbf{V}_{\text{ph}} = 0 \, .$$

If the medium is moving, the disturbance is carried with it.

Substituting (9.13) in (9.8), we obtain the following equations:

$$(\mathbf{k} \cdot \mathbf{u}) \, \mathbf{u}' = 0 \, , \qquad (\mathbf{k} \cdot \mathbf{u}) \, \mathbf{v}' = 0 \, ,$$

$$p' + \rho \, \mathbf{u} \cdot \mathbf{u}' = 0 \, , \qquad p' - V_{\text{s}}^{\,2} \rho' - b \, s' = 0 \, .$$

Since generally $\mathbf{k} \cdot \mathbf{u} \neq 0$, the velocity, magnetic field and gas pressure are undisturbed:

$$\mathbf{v}' = 0 \, , \qquad \mathbf{u}' = 0 \, , \qquad p' = 0 \, . \tag{9.14}$$

The only disturbed quantities are the density and entropy related by the condition

$$\boxed{\rho' = -\frac{b}{V_s^2}\, s'\,.} \tag{9.15}$$

This is why these disturbances are called the **entropy** waves.

They are well known in hydrodynamics.

The meaning of an entropy wave is that regions containing **hotter** but more **rarefied** plasma can exist in a plasma flow.

The entropy waves are only arbitrarily termed **waves**, since their velocity of propagation with respect to the medium is zero.

Nevertheless the entropy waves must be taken into account together with the real waves in such cases as the study of **shock waves** behavior under small perturbations.

Blokhintsev (1945) has considered the passage of small perturbations through a shock in ordinary hydrodynamics.

He came to the conclusion that

> the entropy wave must be taken into account in order to match the linearized solutions at the shock front.

In MHD, the entropy waves are important in the problem of **evolutionarity** of the MHD discontinuities and **reconnecting current layers**.

The entropy waves can be principally essential in astrophysical plasma where plasma motions are not slow, for example in **helioseismology** of the corona of the Sun or another star.

Meanwhile the entropy waves are generally (i.e., with the inclusion of dissipative processes) believed to be **damped** ones.

For this reason it is commonly assumed that the entropy waves may be **ignored**, for example, in the problem of solar coronal heating.

However we shall consider the stability problem for MHD perturbations in an optically thin, perfectly conducting plasma with a cosmic abundance of elements.

It appears that the entropy waves can **grow** exponentially, for example, in **stellar coronae** with the proper allowance for **radiative losses** of energy.

9.2.2 Alfvén waves

The second root of the dispersion equation,

$$\omega_0^2 = (\mathbf{k} \cdot \mathbf{u})^2 \quad \text{or} \quad \omega_0 = \pm\, \mathbf{k} \cdot \mathbf{u}\,, \tag{9.16}$$

corresponds to waves with the phase velocity

$$\boxed{V_{\text{A}} = \pm \frac{B}{\sqrt{4\pi\rho}} \cos\theta\,.} \tag{9.17}$$

Here θ is the angle between the direction of wave propagation $\mathbf{k}/k$ and the ambient field vector $\mathbf{B}_0$ (Fig. 9.1).

In formula (9.17) the value $B = |\, \mathbf{B}_0 \,|$ and $\rho = \rho_0$. These are the **Alfvén** waves.

By substituting (9.16) in the algebraic Equations (9.8) we check that the thermodynamic characteristics of the medium remain unchanged

$$\rho' = 0\,, \qquad p' = 0\,, \qquad s' = 0\,, \tag{9.18}$$

while the perturbations of the velocity and magnetic field are subject to the conditions

$$\mathbf{v}' = \mp\, \mathbf{u}'\,, \qquad \mathbf{u} \cdot \mathbf{u}' = 0\,, \qquad \mathbf{k} \cdot \mathbf{u}' = 0\,. \tag{9.19}$$

Thus the Alfvén waves are the displacements of plasma together with the magnetic field frozen into it.

They are **transversal** with respect to both the field direction and the wave vector as shown in Fig. 9.1.

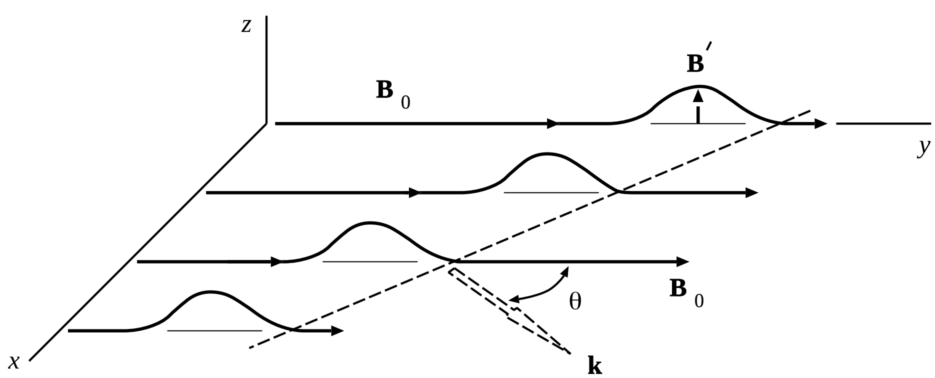


Figure 9.1: The transversal displacements of plasma and magnetic field in the Alfvén wave.

The Alfvén waves have no analogue in hydrodynamics.

They are specific to MHD and were called the **magneto-hydrodynamic** waves.

This term emphasized that they do not change the density of a medium.

The fact that the Alfvén waves are transversal signifies that

> a conducting plasma in a magnetic field has a characteristic **elasticity** resembling that of stretched **strings** under tension.

The **magnetic tension force** is one of the characteristics of MHD.

According to (9.19), the perturbed quantities are related by an energy equipartition:

$$\frac{1}{2}\,\rho\,(v'\,)^2 = \frac{1}{8\pi}\,(B'\,)^2\,. \tag{9.20}$$

Let us note also that

> the energy of Alfvén waves, much like the energy of oscillations in a stretched string, propagates along the field lines only.

Unlike the phase velocity, the **group velocity** of the Alfvén waves (9.12)

$$\mathbf{V}_{\rm gr} = \pm\,\frac{\mathbf{B}}{\sqrt{4\pi\rho}} \tag{9.21}$$

is directed strictly along the magnetic field.

In low density plasmas with a strong field, like the solar corona, the Alfvén speed $V_{\rm A}$ can approach the light speed c (Exercise 9.3).

9.2.3 Magnetoacoustic waves

Equation (9.10) has two other branches – two types of waves defined by a bi-square equation

$$\omega_0^4 - k^2 \left(u^2 + {V_s}^2 \right) \omega_0^2 + k^2 \, {V_s}^2 \, (\mathbf{k} \cdot \mathbf{u})^2 = 0 \, . \tag{9.22}$$

Its solutions are two values of ω_0, which differ in absolute magnitude, corresponding to **two different waves** with the phase velocities V_+ and V_- which are equal to

$$V_\pm^2 = \frac{1}{2} \left[u^2 + {V_s}^2 \pm \sqrt{\left(u^2 + {V_s}^2 \right)^2 - 4 u^2 {V_s}^2 \, \cos^2 \theta} \, \right] . \tag{9.23}$$

These waves are called the **fast** (+) and the **slow** (−) **magnetoacoustic** waves, respectively.

The point is that the entropy does not change in such waves

$$s' = 0 \, , \tag{9.24}$$

as is also the case in a sound wave.

Perturbations of the other quantities can be expressed in terms of the density perturbation

$$p' = {V_s}^2 \rho' \, , \tag{9.25}$$

$$\mathbf{v}' = -\frac{\omega_0}{\rho\, k^2} \left(\frac{k^2 (\mathbf{k} \cdot \mathbf{u})\, \mathbf{u} - \omega_0^2\, \mathbf{k}}{\omega_0^2 - (\mathbf{k} \cdot \mathbf{u})^2} \right) \rho' \, , \tag{9.26}$$

$$\mathbf{u}' = \frac{\omega_0^2}{\rho\, k^2} \left(\frac{k^2\, \mathbf{u} - (\mathbf{k} \cdot \mathbf{u})\, \mathbf{k}}{\omega_0^2 - (\mathbf{k} \cdot \mathbf{u})^2} \right) \rho' \, . \tag{9.27}$$

Formulae (9.26) and (9.27) show that the magnetoacoustic waves are **neither** longitudinal **nor** transversal.

Perturbations of the velocity and magnetic field, $\mathbf{v}'$ and $\mathbf{u}'$, as differentiated from the Alfvén wave, lie in the $(\mathbf{k}, \mathbf{B}_0)$ plane in Fig. 9.1.

They have components both in the direction of the wave propagation $\mathbf{k}/k$ and in the perpendicular direction.

The perturbation of magnetic pressure $B^2/8\pi$ may be written in the form

$$p'_{\mathrm{m}} = \left(\frac{V_{\pm}^2}{{V_s}^2} - 1 \right) p' \, . \tag{9.28}$$

Therefore for the **fast** wave, by virtue of that $V_+^2 > {V_s}^2$, the perturbation of magnetic pressure p'_{m} is of the same sign as that of gas pressure p'.

> The magnetic pressure and the gas pressure are added in the fast magnetoacoustic wave. The wave propagates faster, since the **effective elasticity** of the plasma is greater.

A different situation arises with a **slow** magnetoacoustic wave.

In this case $V_{-}^{2} < V_{s}^{2}$ and $p'_{\rm m}$ is **opposite** in sign to p'.

Magnetic and gas pressure deviations partially compensate each other.

That is why such a **slow** wave propagates **slowly**.

9.2.4 The phase velocity diagram

The dependence of the wave velocities on the angle θ between the undisturbed field $\mathbf{B}_0$ and the wave vector $\mathbf{k}$ is demonstrated in a polar diagram – the phase velocity diagram.

In Fig. 9.2, the **radius-vector length** from the origin of the coordinates to a curve is proportional to the corresponding phase velocity.

The horizontal axis corresponds to the direction of the magnetic field.

As $\theta \rightarrow 0$, the fast magnetoacoustic wave V_+ transforms to the usual sound one V_s if

$$V_s > V_{\rm A\,\|} = \frac{B}{\sqrt{4\pi\rho}} \equiv u_{\rm A} \tag{9.29}$$

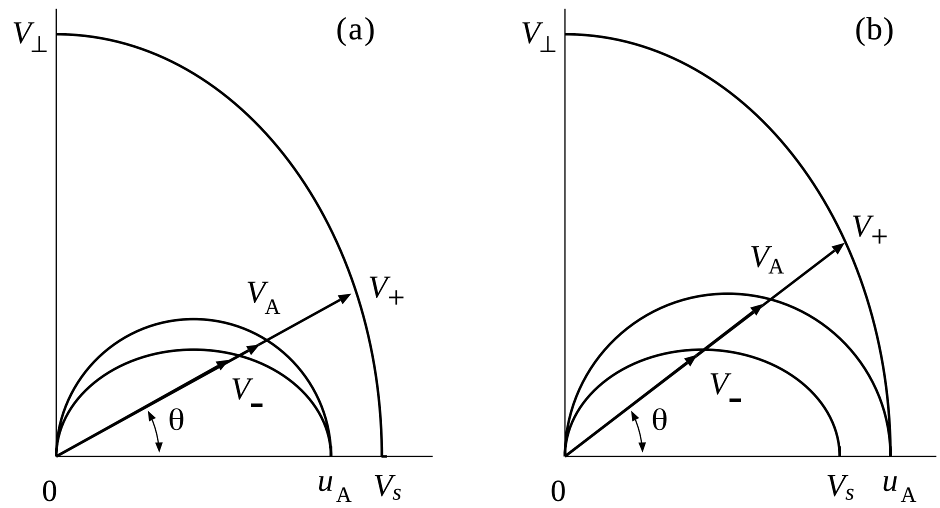


Figure 9.2: The phase velocities of MHD waves versus the angle θ for the two cases: (a) $u_{\mathrm{A}} < V_s$ and (b) $u_{\mathrm{A}} > V_s$.

in Fig. 9.2a or to the Alfvén wave if $V_s < u_{\mathrm{A}}$ in Fig. 9.2b.

For $\theta \to \pi/2$, the propagation velocities of the Alfvén and slow waves approach zero.

As this takes place, both waves convert to the **weak tangential** discontinuity in which disturbances of velocity and magnetic field are parallel to the front plane.

As $\theta \to \pi/2$, the fast magnetoacoustic wave velocity tends to

$$V_\perp = \sqrt{u_{\mathrm{A}}^2 + V_s^{\,2}}\,. \tag{9.30}$$

In the **strong** field limit ($V_{\mathrm{A}\,\|}^{\;2} \gg V_s^{\,2}$) the diagram for the fast magnetoacoustic wave becomes practically isotropic as shown in Fig. 9.3.

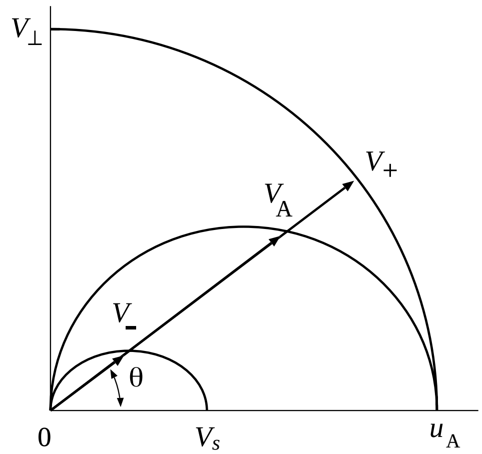


Figure 9.3: The phase velocity diagram for the MHD waves in an ideal plasma with a strong magnetic field.

Such a wave may be called the '**magnetic sound**' wave since its phase velocity

$$V_+ \approx V_{A\,\|} \equiv u_A$$

is almost independent of the angle θ.

> Generally the sound speed is the minimum velocity of disturbance propagation in ordinary hydrodynamics. By contrast, there is **no minimum velocity** in MHD.

This property is of fundamental importance in study of the **principal questions** related to **discontinuous flows** of astrophysical plasma.

The first of these questions is what kinds of discontinuities can really exist?

MHD waves produce a lot of effects in astrophysical plasma. The fast magnetoacoustic wave **turbulence** can presumably accelerate electrons in solar flares.

The heavy ions observed in interplanetary space after impulsive flares can result from **stochastic acceleration** by the cascading Alfvén wave turbulence.

9.3 Dissipation of Alfvén waves

9.3.1 Small damping of Alfvén waves

We shall start by treating a plane Alfvén wave propagating along a uniform field $\mathbf{B}_0$; so the angle $\theta = 0$ in Fig. 9.1.

Perturbations of the magnetic field and the velocity are parallel to the z axis:

$$\mathbf{B}' = \{\, 0,\, 0,\, b\,(t, y)\,\}\,, \qquad \mathbf{v}' = \{\, 0,\, 0,\, v\,(t, y)\,\}\,.$$

In general, the damping effects for such a wave are determined by viscosity and conductivity.

Let us consider, first, only the uniform finite **conductivity** σ.

We obtain the extended equation of the wave type with a dissipative term:

$$\frac{\partial^2 b}{\partial t^2} = u_{\mathrm{A}}^2 \, \frac{\partial^2 b}{\partial y^2} + \nu_{\mathrm{m}} \, \frac{\partial^3 b}{\partial^2 y \, \partial t}\,. \tag{9.31}$$

Here $u_{\mathrm{A}} = V_{\mathrm{A}\,\|}$ and ν_{m} is the magnetic diffusivity.

In the case of **infinite conductivity**, (9.31) is reduced to the wave equation.

Let us suppose that the conductivity is **finite**.

We suppose further that the small perturbations are functions of t and y only:

$$b\,(t,y) = b_0\,\exp\,(\,\mathrm{i}\,\omega t + \alpha y)\,,$$

$$v\,(t,y) = v_0\,\exp\,(\,\mathrm{i}\,\omega t + \alpha y)\,. \tag{9.32}$$

Here ω, α, b_0, and v_0 are constants, all of which except ω may be complex numbers.

Substituting (9.32) in (9.31) gives us the dispersion equation

$$\omega^2 + \left(u_{\mathrm{A}}^2 + \mathrm{i}\,\nu_{\mathrm{m}}\,\omega\right)\alpha^2 = 0 \tag{9.33}$$

or

$$\alpha = \pm\,\mathrm{i}\,\frac{\omega}{u_{\mathrm{A}}}\left(1 + \mathrm{i}\,\frac{\nu_{\mathrm{m}}\,\omega}{u_{\mathrm{A}}^2}\right)^{-1/2}. \tag{9.34}$$

For **small damping**

$$\alpha = \pm\left(\frac{\nu_{\mathrm{m}}\,\omega^2}{2\,u_{\mathrm{A}}^3} + \mathrm{i}\,\frac{\omega}{u_{\mathrm{A}}}\right). \tag{9.35}$$

The distance l_d in which the amplitude of the wave is reduced to $1/e$ is the inverse value of the real part of α.

Thus we have

$$l_d = \frac{2\,u_{\mathrm{A}}^3}{\nu_{\mathrm{m}}\,\omega^2} = \frac{8\pi\sigma u_{\mathrm{A}}^3}{\omega^2 c^2} = \frac{2\sigma u_{\mathrm{A}}}{\pi c^2}\,\lambda^2, \tag{9.36}$$

where

$$\lambda = 2\pi\,u_{\mathrm{A}}/\omega$$

is the wave length.

The **short waves suffer more damping** than do the long waves.

Since we treat the dissipative effects as small, the expression (9.36) is valid if $\lambda \ll l_d$.

Thus we write

$$b\,(t, y) = b_0 \, \exp\left(-\frac{y}{l_d}\right) \exp\left[\mathrm{i}\,\omega\left(t - \frac{y}{u_{\mathrm{A}}}\right)\right], \tag{9.37}$$

$$v\,(t, y) = v_0 \, \exp\left(-\frac{y}{l_d}\right) \exp\left[\mathrm{i}\,\omega\left(t - \frac{y}{u_{\mathrm{A}}}\right)\right] \tag{9.38}$$

with

$$v_0 = u_{\mathrm{A}} \frac{b_0}{B_0}\left(1 - \mathrm{i}\,\frac{\nu_{\mathrm{m}}\,\omega}{2\,u_{\mathrm{A}}^2}\right). \tag{9.39}$$

The imaginary part indicates the **phase shift** of the velocity v in relation to the magnetic perturbation field b. Therefore

$$v\,(t, y) = u_{\mathrm{A}} \frac{b_0}{B_0} \, \exp\left(-\frac{y}{l_d}\right) \exp\left\{\mathrm{i}\left[\omega\left(t - \frac{y}{u_{\mathrm{A}}}\right) - \varphi\right]\right\}, \tag{9.40}$$

where

$$\varphi = \frac{\nu_{\mathrm{m}}\,\omega}{2\,u_{\mathrm{A}}^2} = \frac{\omega\, c^2}{8\pi\sigma u_{\mathrm{A}}^2} = \frac{\omega\, c^2 \rho}{2\sigma B_0^2}\,.$$

So the existence of Alfvén waves requires an external field B_0 enclosed between **two limits**.

> The magnetic field should be strong enough to make the **damping effects small** ($l_d \gg \lambda$) but yet weak enough to keep the Alfvén speed well below the velocity of light,

because otherwise the wave becomes an ordinary electromagnetic wave.

In optical and radio frequencies it is not possible to satisfy both conditions.

However longer periods often observed in astrophysical plasma leave a **wide range** between both limits so that Alfvén waves may easily exist.

One of favorable sites for excitation of MHD waves is the solar atmosphere.

The chromosphere and corona are highly inhomogeneous media supporting a variety of filamentary structures in the form of arches and loops.

The foot points of these structures are anchored in the poles of the photospheric magnetic fields.

They undergo a continuous twisting and turning due to convective motions in the subphotospheric layers.

This twisting and turning excite MHD waves.

The waves then dissipate and heat the corona.

Presumably this energy is enough to explain coronal heating.

9.3.2 Slightly damped MHD waves

The damping effects due to a finite **conductivity** σ and due to a kinematic **viscosity** $\nu = \eta/\rho$ can be included in a general treatment of MHD waves of small amplitudes.

Well developed waves are the waves that travel at least a few wave lengths before they lose a considerable fraction of their energy if the two dimensionless parameters

$$p_{\nu} = \frac{\omega\, \nu}{c^2} \quad \text{and} \quad p_{\nu_{\rm m}} = \frac{\omega\, \nu_{\rm m}}{c^2}\,, \tag{9.41}$$

that characterize two dissipative processes, are much smaller than the two small dimensionless parameters

$$p_{\,s} = \frac{V_s^2}{c^2} \quad \text{and} \quad p_{\rm A} = \frac{u_{\rm A}^2}{c^2}\,, \tag{9.42}$$

that characterize the propagation speeds of undamped waves.

For **Alfvén wave**, we find the **damping length**, i.e. the distance l_d and the **damping time** τ_d:

$$l_d = \frac{u_{\rm A}^3}{\omega^2\,(\nu + \nu_{\rm m})}\,, \tag{9.43}$$

$$\tau_d = \frac{l_d}{u_{\rm A}} = \frac{u_{\rm A}^2}{\omega^2\,(\nu + \nu_{\rm m})}\;. \tag{9.44}$$

This shows that, if dissipative effects are small,

> the relative importance of resistivity and viscosity as damping effects in Alfvén wave is independent of frequency ω.

So the **high** frequency waves have a **short** damping length and time.

The **magnetoacoustic waves**, being compressional, have an additional contribution to their damping rate from **compressibility** of the plasma.

9.4 Stability of plasma-compressing waves

Let us consider the stability problem for small MHD perturbations in an **optically thin** plasma with a cosmic abundance of elements.

9.4.1 Derivation of the dispersion equation

Disregarding the damping effects of ordinary and magnetic viscosities, considered above, let us write the set of MHD equations as

$$\frac{d\rho}{dt} + \rho \, \mathrm{div} \, \mathbf{v} = 0 \,,$$

$$\rho \, \frac{d\mathbf{v}}{dt} + \nabla \left(p + \frac{B^2}{8\pi} \right) - (\mathbf{B} \cdot \nabla) \, \mathbf{B} \, \frac{1}{4\pi} = 0 \,,$$

$$\rho \, T \, \frac{ds}{dt} = \frac{1}{\gamma - 1} \, \frac{dp}{dt} - \frac{\gamma}{\gamma - 1} \, \frac{p}{\rho} \, \frac{d\rho}{dt} =$$

$$= \nabla_{\parallel} \cdot \left(\kappa_{\parallel} \nabla_{\parallel} T \right) + \nabla_{\perp} \cdot \left(\kappa_{\perp} \nabla_{\perp} T \right) - \mathcal{L}(\rho, T) + \mathcal{H} \,, \quad (9.45)$$

$$\frac{d\mathbf{B}}{dt} + \mathbf{B} \, \mathrm{div} \, \mathbf{v} - (\mathbf{B} \cdot \nabla) \, \mathbf{v} = \mathbf{0} \,,$$

$$\operatorname{div} \mathbf{B} = 0 \,,$$

$$p = \frac{R}{\mu}\, \rho T \,.$$

Here

s is the entropy per unit mass,

$\gamma = c_p / c_v$ is the ratio of the specific heats at constant pressure and constant volume, and

$\mathcal{L}$ is the rate of **energy losses** through the optically thin plasma radiation per unit volume measured in erg $\mathrm{s}^{-1}\,\mathrm{cm}^{-3}$ (see formula (6.22) and Fig. 6.1).

The heat conduction along ($\kappa_{\parallel}$) and across ($\kappa_{\perp}$) the field is determined, respectively, by electrons and, for a completely ionized hydrogen plasma, by protons.

The existence of a steady state in such a plasma implies the presence of a steady uniform heating source whose power $\mathcal{H}$ is equal to the rate of radiative energy losses.

Let us represent all of the physical variables as

$$f(\mathbf{r},\, t) = f_0 + f^{\,\prime}(\mathbf{r},\, t) \,.$$

Since the medium and the field are assumed to be uniform, we seek a solution in the form of plane waves

$$f^{\,\prime}(\mathbf{r},\, t) = f_1 \, \exp i \, (\, \mathbf{k} \cdot \mathbf{r} - \omega t) \,. \tag{9.46}$$

We obtain a set of linear algebraic equations with constant coefficients:

$$\omega \rho_1 - \rho_0 \, (\, \mathbf{k} \cdot \mathbf{v}_1) = 0 \,, \tag{9.47}$$

$$\omega\rho_0\,\mathbf{v}_1 - \mathbf{k}\left(p_1 + \frac{\mathbf{B}_0\cdot\mathbf{B}_1}{4\pi}\right) + (\,\mathbf{k}\cdot\mathbf{B}_0)\,\frac{\mathbf{B}_1}{4\pi} = 0\,, \qquad (9.48)$$

$$\frac{(-i)\,\omega}{\gamma-1}\left(p_1 - c_s^2\rho_1\right) + Q_\rho\,\rho_1 + Q_{T}\,T_1 + \mathcal{K}\,T_1 = 0\,, \qquad (9.49)$$

$$\omega\,\mathbf{B}_1 - \mathbf{B}_0\,(\,\mathbf{k}\cdot\mathbf{v}_1) + (\,\mathbf{k}\cdot\mathbf{B}_0)\,\mathbf{v}_1 = 0\,, \qquad (9.50)$$

$$(\,\mathbf{k}\cdot\mathbf{B}_1\,) = 0\,, \qquad (9.51)$$

$$\frac{p_1}{p_0} = \frac{\rho_1}{\rho_0} + \frac{T_1}{T_0}\,. \qquad (9.52)$$

Here $c_s^2 = \gamma\,p_0/\rho_0$ is the sound speed squared; the coefficients

$$Q_\rho = \left.\frac{\partial\mathcal{L}}{\partial\rho}\right|_T\,, \qquad Q_{T} = \left.\frac{\partial\mathcal{L}}{\partial T}\right|_\rho \qquad (9.53)$$

and

$$\mathcal{K} = \kappa_{\parallel}\,k_{\parallel}^2 + \kappa_{\perp}\,k_{\perp}^2 \qquad (9.54)$$

characterize the plasma **emissivity** and anisotropic **heat conduction**.

The dispersion equation can be written as

$$(\,\mathbf{k}\cdot\mathbf{v}_1)\left[-\,\omega\,\frac{4\pi}{k^2}\,(\,\Sigma\,k^2 + \omega\rho_0\,) + B_0^2 + \frac{\Sigma}{\omega\rho_0}\,(\,\mathbf{k}\cdot\mathbf{B}_0)^2\right] =$$

$$= 0\,, \qquad (9.55)$$

where the function

$$\Sigma\,(\omega) = \qquad (9.56)$$

$$= \frac{(-i)\,\omega\,\rho_0\,c_s^2 - (\gamma-1)\,\rho_0^2\,Q_\rho + (\gamma-1)\,T_0\,(\,Q_{T} + \mathcal{K})}{\omega\,[\,i\,\omega\rho_0 - \mu\,(\gamma-1)\,(\,Q_{T} + \mathcal{K})/R\,]}\;\rho_0\,.$$

9.4.2 The instability of entropy waves

In strong magnetic field, the entropy waves grow exponentially if (Somov et al., 2007):

$$Q_{_T} + \mathcal{K} < 0\,. \tag{9.57}$$

Since the coefficient $\mathcal{K} > 0$ by definition, the **heat conduction** has a stabilizing effect on the entropy waves.

It thus follows that the instability can manifest itself only at negative values of the derivative of the rate of radiative losses,

$$\mathcal{L} \simeq -\, n^2\, q(T)\,, \;\; \mathrm{erg\ s^{-1}\, cm^{-3}}\,,$$

(see the function $q(T)$ in Fig. 6.1) with respect to the temperature.

The logarithmic derivative of the cooling function,

$$\alpha(T) = \mathrm{d} \ln q(T) \,/\, \mathrm{d} \ln T\,,$$

is shown in Fig. 9.4.

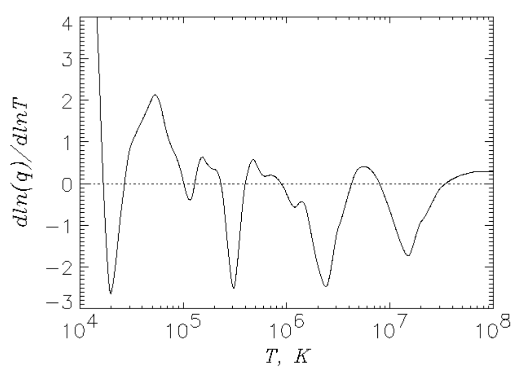


Figure 9.4: Logarithmic derivative of the cooling function q with respect to the temperature.

Now we return to the general case.

The exact solutions of the dispersion equation show that there are temperature ranges in which the entropy waves grow exponentially.

The growth time τ is plotted against the temperature T in Fig. 9.5 for three fixed plasma densities: $n_1 = 10^9$, $n_2 = 10^{10}$, $n_3 = 10^{11}$ cm^{-3}.

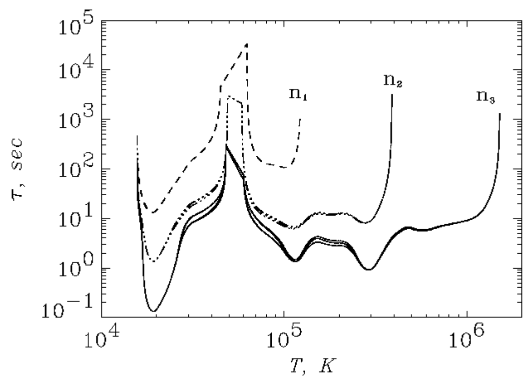


Figure 9.5: The growth time for entropy wave instability vs. temperature at fixed plasma densities.

A weak dependence on the magnetic field enters into the calculations with formulae for the heat conductivity

$$\kappa_{\parallel} \simeq 9 \times 10^{-7} T^{5/2}, \qquad \frac{\kappa_{\perp}}{\kappa_{\parallel}} \simeq 2 \times 10^{-11} \frac{n^2}{T^3 B^2} ,$$

where the heat conductivity is measured in CGS units and the field B is measured in Gauss.

The calculations were performed for a wide range of field strengths, $B = (10^{-1} - 10^4)$ G.

This manifested itself in Fig. 9.5 as a slight "stratification" of the curves for the instability growth time τ at the plasma densities $n_2 = 10^{10}$ and $n_3 = 10^{11}$ cm^{-3}.

Here we took the angle θ to be $\pi/4$ and chose the wavelength to be $\lambda = 10^8$ cm.

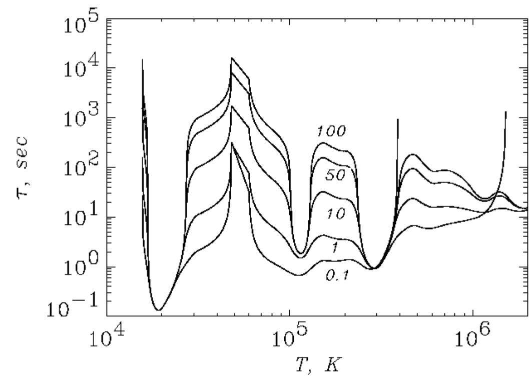


Figure 9.6: The growth time as a function of temperature and the wavelength λ indicated as a number near the curves (the unit of length is 10^8 cm).

The wavelength dependence of the growth time τ is shown in Fig. 9.6.

The magnetic field strength is $B_0 = 1000$ G.

> The growth time for entropy waves in stellar coronae can vary over a wide range: **from tenths** of a second **to tens** minutes.

The higher the plasma density, the faster the instability growth.

Two "main" minima (regions of **strong instability**) are present at $T_1 \approx 2 \times 10^4$ K and $T_2 \approx 3 \times 10^5$ K.

The instability mechanism is simple.

In the temperature regions of a rapid decrease in the radiative loss function with temperature, a small decrease in temperature causes a large increase in the rate of radiative energy losses.

Conversely a small increase in temperature is accompanied by a decrease in the rate of radiative plasma cooling.

As a result, small perturbations grow rapidly.

Clearly, in general,

> the entropy waves can be unstable in any compressible optically thin medium whose **radiative cooling** depends on the temperature in the form of one or more **sharp maxima**.

In astrophysical and laboratory plasmas, these maxima are produced by the radiation of a small admixture of heavy ions.

However, in contrast to the astrophysical plasma, the approximation of an ideal conductivity is **not** applicable in the laboratory plasma.

The **Joule heating** becomes the dominant heating mechanism.

The dispersion equation includes the derivative with respect to the temperature not only the bulk radiation intensity but also of the **plasma conductivity**.

This leads to instabilities and complex dynamics of plasma, for example, in emitting Z-pinches (Imshennik and Bobrova, 1997).

9.4.3 The damping of magnetoacoustic waves

In contrast to the entropy waves, the magnetoacoustic waves are damped.

At the assumed physical model parameters consistent with our views of the solar and stellar coronae, the damping decrement is

$$d = \frac{\text{Im }\omega}{\text{Re }\omega} < 0\,.$$

For **fast** magnetoacoustic waves, Fig. 9.7 presents the solutions to the dispersion equation at the same density, angle θ and the magnetic field as those in Fig. 9.6.

The damping decrement d in the temperature range $10^4 \leq T \leq 2 \times 10^6$ K is very small: $-d < 10^{-6}$.

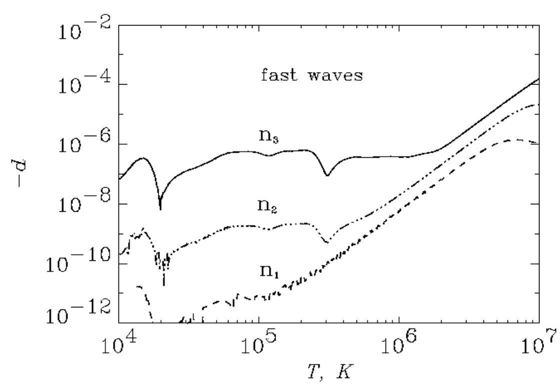


Figure 9.7: Damping decrement of fast waves as a function of the temperature and density for the same parameters as those in Fig. 9.6.

For **slow** magnetoacoustic waves, the damping decrement is several orders of magnitude larger (Fig. 9.8).

At $T_2 \approx 3 \times 10^5$ K, the damping decrement lies within the range $\sim 0.05 < -d < 0.16$.

This corresponds to a decrease in the wave amplitude by a factor of e in a time τ equal to $\sim 3-1$ wave periods $\tau_\omega = 2\pi/\operatorname{Re}\omega$.

As the temperature decreases to $T \approx T_1 \approx 2 \times 10^4$ K, the damping decrement increases to values of the order of several seconds.

> Thus the damping decrement for slow waves is 4–6 orders of magnitude larger than that for fast waves.

What is the cause of such a large difference? –

In a **fast** magnetoacoustic wave, the sign of the change in magnetic pressure is the same as that of the change in gas pressure, i.e., the magnetic and gas pressures add up.

Where the plasma density increases, the magnetic pressure also increases, preventing the plasma compression.

The effective plasma elasticity for a fast wave is larger.

The picture is different in a **slow** wave.

In this case, the changes in magnetic and gas pressures have opposite signs.

> Where the plasma density increases, the magnetic field decreases, without preventing the plasma from compressing. As a result, the radiative energy losses grow faster than that of fast waves.

For this reason, the radiative damping of slow waves is much stronger than that of fast waves.

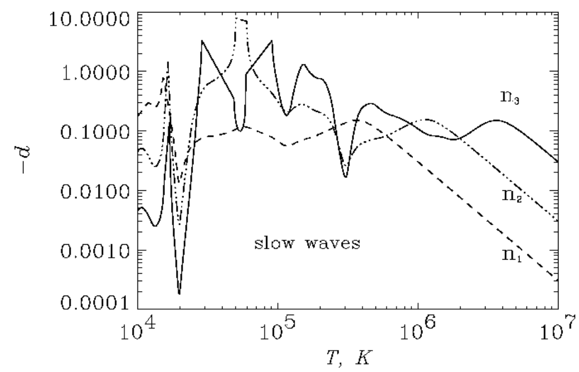


Figure 9.8: Damping decrement of slow magnetoacoustic waves.

One would think that slow waves resemble more closely the ordinary sound and that the excitation of thermal instability might be expected for them (Field, 1965).

Instead, the calculations demonstrate only rapid damping of slow waves.

However there is no contradiction in this fact.

The point is that the magnetoacoustic waves are neither longitudinal nor transverse.

Moreover, the longitudinal components are not independent of the transverse ones.

For this reason, the stabilizing effect of a strong magnetic field dominates over the excitation of oscillations by **thermal instability**.

Of course, the damping of oscillations due to the radiative energy losses is present and turns out to be strong for slow waves.

9.5 MHD oscillations in the solar corona

In the corona, the low-frequency MHD oscillations can be studied well almost at all wavelengths.

Most of these oscillations are commonly interpreted as standing oscillations of various types in coronal magnetic loops.

Meanwhile the oscillations of coronal loops observed from **TRACE** in EUV are, as a rule, **damped rapidly**.

The ratio of the characteristic damping time τ_d to the oscillation period τ_ω is

$$\tau_d / \tau_\omega = 1.8 \pm 0.8$$

in the range of periods

$$\tau_\omega = 317 \pm 114 \text{ s}.$$

Such **rapid damping** of the MHD oscillations is difficult to explain.

The **current models** of damping mechanisms consider the following factors:

(a) nonideality of the solar plasma, i.e. the presence of electric **resistivity**, **viscosity**, and **heat conduction** and **radiative losses** of energy;

(b) the **escape of waves** from the magnetic loop through its side boundaries into corona and through the loop foot-points into the chromosphere;

(c) **phase mixing** in regions where the magnetic field and the plasma are nonuniform;

(d) resonant absorption.

The first two factors are believed to be negligible and the last two factors are believed to be the main ones.

Since the coronal oscillations are damped rapidly, the question of their **excitation mechanism** also arises.

If the oscillations grew rapidly, there would be no problem: low-amplitude perturbations would sufficient.

However the formal **number of oscillations** is, in average,

$$N_{obs} = \frac{\tau_d}{\tau_\omega} \approx 2\,. \tag{9.58}$$

The current models just **postulate** instantaneous excitation of large-amplitude MHD oscillations in an isolated magnetic flux tube.

Why rapidly damped oscillations are seen best in a **small group** of loops precisely in **EUV** radiation is probably a key question.

Contrary to popular belief, the answer may be simple.

> Where the rate of energy losses via radiation is at a maximum (at $T \sim 10^5$ K), the brightness of the oscillating loops is at a maximum (in EUV) and the oscillations are damped most rapidly.

The calculations presented above show that the predicted number of oscillations for **slow** magnetoacoustic waves is

$$N_{th} = \frac{\tau}{\tau_\omega} \approx 1 - 3 \tag{9.59}$$

in the range of temperatures corresponding to the second main maximum, i.e. $T_2 \approx 3 \times 10^5$ K.

Moreover, at the plasma density $n_1 = 10^9\,\text{cm}^{-3}$, the damping decrement remains in the same range as the temperature decreases to $T \approx T_1 \approx 2 \times 10^4$ K.

For the 11 events with oscillating coronal loops that were studied in detail by Aschwanden et al. (2003), the plasma density inside the loops was, on average, $n_{int} = (1.4 \pm 0.7) \times 10^9\,\text{cm}^{-3}$.

Consequently

$$n_{int} \approx n_1$$

and

$$N_{th} \approx N_{obs}\,.$$

Thus we suggest that

> the observed oscillations are a manifestation of the rapid damping of slow magnetoacoustic waves due to the radiative energy losses in the corona.

The **excitation mechanism** of magnetoacoustic waves should be studied further.

The rapid growth of **entropy waves** calculated above must probably be taken into account to explain this phenomenon.

Clearly, we cannot ignore the fast plasma flows (particularly those from **reconnecting current layers**) and, hence, the growing entropy waves transferred by these flows in the corona.